Math 1210: Calculus I
The Chain Rule

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.5

Derivatives and rules

We have several tools for computing derivatives:

- $(c_1 f(x) + c_2 g(x))' = c_1 f'(x) + c_2 g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
- $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$
- $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$.

All these rules allow us to compute derivatives quite well.

There is one simple example that's still difficult for us.

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Example

Compute the derivative of $f(x) = (x^2 + 3)^{45}$.

Composite functions

The core challenge with the previous example is that $f(x)$ was a composite function:

$f(x) = (x^2+3)^{45}$

$$f(x) = g(k(x)), \qquad g(x) = x^{45}, \qquad k(x) = x^2 + 3$$

Note that both $g'(x)$ and $k'(x)$ are simple to compute, but f' is not.

$g'(x) = 45x^{44}$ $\qquad$ $k'(x) = 2x + 0$

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The tool we are missing is something that allows us to differentiate *through* function composition.

The Chain Rule will allow us to do this.

Increments again

The chain rule idea can be motivated through increments.
Recall:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\Delta f}{\Delta x},$$

where Δf and Δx are increments, $h = (x+h) - x$

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Of course the whole point is that differentiating f was hard, but differentiating g and k was easy.

We can introduce g and k by multiplying and dividing by another increment.

I.e. what is $f'(x)$ in terms of $g'(x)$ and $k'(x)$?

Intuition through increments

(Recall $f(x) = g(k(x))$)

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x},$$

$$= \lim_{h \to 0} \frac{g(k(x+h)) - g(k(x))}{h}$$

"$\frac{\Delta g}{\Delta k} \cdot \frac{\Delta k}{\Delta x}$"

$$= \lim_{h \to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h) - k(x)} \underbrace{\frac{k(x+h) - k(x)}{h}}_{k'(x) \text{ as } h \to 0}$$

The second term would turn into the derivative of k.

What about the first term?

Intuition through increments

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\Delta f}{\Delta x}, \\ &= \lim_{h\to 0} \frac{g(k(x+h)) - g(k(x))}{h} \\ &= \lim_{h\to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h)-k(x)} \underbrace{\frac{k(x+h)-k(x)}{h}}_{k'(x) \text{ as } h \to 0} \end{aligned}$$

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What about the first term?

$$\lim_{h\to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h)-k(x)} \text{``}=\text{''} \lim_{h\to 0} \frac{\Delta g}{\Delta k} = g'(k(x))$$

There is some new notation we haven't really seen before:

$$g'(k(x)) \text{ means the function } g' \text{ evaluated at } k(x).$$

The chain rule

Putting things together:

$f(x)$
$= g(k(x))$

$$f'(x) = \lim_{h \to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h) - k(x)} \, \frac{k(x+h) - k(x)}{h} = g'(k(x))k'(x)$$

Informally, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{\Delta f}{\Delta x} = \frac{\Delta g(k(x))}{\Delta x} = \frac{\Delta g}{\Delta k} \, \frac{\Delta k}{\Delta x} \approx \frac{\mathrm{d}g}{\mathrm{d}k}\Big|_{k=k(x)} \frac{\mathrm{d}k}{\mathrm{d}x}$$

The chain rule

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$$f'(x) = \lim_{h \to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h) - k(x)} \, \frac{k(x+h) - k(x)}{h} = g'(k(x))k'(x)$$

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$$\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{\Delta f}{\Delta x} = \frac{\Delta g(k(x))}{\Delta x} = \frac{\Delta g}{\Delta k} \frac{\Delta k}{\Delta x} \approx \left.\frac{\mathrm{d}g}{\mathrm{d}k}\right|_{k=k(x)} \frac{\mathrm{d}k}{\mathrm{d}x}$$

Formally, we have motivated the chain rule:

Theorem (The Chain Rule)

Given differentiable functions $k(x)$ and $g(x)$, suppose that $f(x) = g(k(x))$. Then:

$$f'(x) = g'(k(x))\, k'(x)$$

This rule is nontrivial, but mastery allows you to differentiate almost any function you can write down.

$f(x) = 2^x$ $f(x) = x^x$

Example

Compute the derivative of $f(x) = (x^2+3)^{45}$.
(Ans: $f'(x) = 90x(x^2+3)^{44}$.)

$g(x) = x^{45} \qquad k(x) = x^2+3$

$f(x) = g(k(x)) = g(x^2+3) = (x^2+3)^{45}$

Chain rule: $f'(x) = g'(k(x)) \cdot k'(x)$

$g'(x) = 45x^{44}, \qquad k'(x) = 2x$

$f'(x) = g'(k(x)) \cdot k'(x) = g'(x^2+3) \cdot 2x = 45(x^2+3)^{44} \cdot 2x$

$= 90x(x^2+3)^{44}$

Example

Compute the derivative of $f(x) = \sin(x^2)$.
(Ans: $f'(x) = 2x\cos(x^2)$.)

$g(x) = \sin x \qquad k(x) = x^2$

$f(x) = g(k(x))$

Chain rule: $f'(x) = g'(k(x)) \cdot k'(x)$

$g'(x) = \cos x, \quad k'(x) = 2x$

$f'(x) = g'(k(x)) \cdot k'(x) = g'(x^2) \cdot 2x = \cos(x^2) \cdot 2x = 2x\cos(x^2)$

Example

Compute the derivative of $f(x) = \tan^4 x$.
(Ans: $f'(x) = 4(\tan^3 x)(\sec^2 x)$.)

$g(x) = x^4, \quad k(x) = \tan x$

$f(x) = g(k(x)) \Longrightarrow f'(x) = g'(k(x)) \cdot k'(x)$

$g'(x) = 4x^3 \quad , \quad k'(x) = \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$= \frac{1}{\cos^2 x} = \sec^2 x$

$f'(x) = g'(k(x)) \cdot k'(x) = g'(\tan x) \cdot \sec^2 x = 4(\tan x)^3 \cdot \sec^2 x$

What if we define $g(x) = \tan x$, $k(x) = x^4$?

$$g(k(x)) = \tan(x^4) \neq \tan^4 x = f(x)$$

$$k(g(x)) = (\tan x)^4 = f(x)$$

$$f'(x) = k'(g(x)) \cdot g'(x)$$

$$f'(x) \cancel{=} g'(k(x)) \cdot k'(x)$$

Example

Compute the derivative of $g(t) = \left(\frac{t^4+3t+2}{t^2+1}\right)^{12}$

(Ans: $g'(t) = 12\left(\frac{t^4+3t+2}{t^2+1}\right)^{11} \frac{(4t^3+3)(t^2+1)-2t(t^4+3t+2)}{(t^2+1)^2}$)

$g(t) = h(k(t))$, $\quad h(t) = t^{12}$, $\quad k(t) = \frac{t^4+3t+2}{t^2+1}$

$h'(t) = 12t^{11}$ $\quad k'(t) = \frac{[t^4+3t+2]'\cdot(t^2+1) - [t^4+3t+2]\cdot(t^2+1)'}{(t^2+1)^2}$

$= \frac{(4t^3+3)(t^2+1)-(t^4+3t+2)\cdot 2t}{(t^2+1)^2}$

$$g'(t) = h'(k(t)) \cdot k'(t)$$

$$= h'\left(\frac{t^4+3t+2}{t^2+1}\right) \cdot \left[\frac{(4t^3+3)(t^2+1)-(t^4+3t+2)\cdot 2t}{(t^2+1)^2}\right]$$

$$= 12\left[\frac{t^4+3t+2}{t^2+1}\right]^{11}\left[\frac{(4t^3+3)(t^2+1)-2t(t^4+3t+2)}{(t^2+1)^2}\right]$$

Example

Compute the derivative of $q(s) = \cos(\sin s^9)$
(Ans: $q'(s) = -9s^8(\sin(\sin s^9))(\cos s^9)$.)

$\cos x^2 \neq \cos^2 x$

$f(s) = \cos(s) \qquad g(s) = \sin(s^9)$

$q(s) = f(g(s)) \implies q'(s) = \underline{f'(g(s))} \cdot g'(s)$

$= -\sin(\sin(s^9)) \cdot g'(s)$

$g(s) = \sin(s^9) = h(k(s)), \quad h(s) = \sin s, \quad k(s) = s^9$

$$\Rightarrow g'(s) = \underline{h'(k(s)) \cdot k'(s)} = \cos(s^9) \cdot 9s^8$$

$$q'(s) = -\sin(\sin(s^9)) \cdot g'(s)$$

$$= -\sin(\sin(s^9)) \cdot \cos(s^9) \cdot 9s^8$$

Here: $q(s) = f(h(k(s)))$

$$\Rightarrow q'(s) = \underline{f'(h(k(s)))} \cdot \underline{h'(k(s)) \cdot k'(s)}$$

$$\frac{d}{dx} \sin x = \frac{d}{dx} f(g(x)), \qquad f(x) = \sin x, \quad g(x) = x$$

$$= f'(g(x)) \cdot g'(x) \qquad f'(x) = \cos x \quad g'(x) = 1$$

$$= \cos(x) \cdot 1$$

The chain rule

The motivation for "chain rule" as a name can be understood as follows:

If $f(x) = a(b(c(x)))$, where a, b, and c are all differentiable, then,

$$f'(x) = a'(b(c(x))) \cdot b'(c(x)) \cdot c'(x),$$

and one can imagine how this might work for an arbitrary number of function compositions by chaining together individual derivatives.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.