Math 1210: Calculus I
The Chain Rule

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.5

Derivatives and rules

We have several tools for computing derivatives:

- $(c_1 f(x) + c_2 g(x))' = c_1 f'(x) + c_2 g'(x)$
- $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g^2(x)}$
- $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$
- $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$.

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Example

Compute the derivative of $f(x) = (x^2 + 3)^{45}$.

Composite functions

The core challenge with the previous example is that $f(x)$ was a composite function:

$$f(x) = g(k(x)), \qquad g(x) = x^{45}, \qquad k(x) = x^2 + 3$$

Note that both $g'(x)$ and $k'(x)$ are simple to compute, but f' is not.

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The tool we are missing is something that allows us to differentiate *through* function composition.

The Chain Rule will allow us to do this.

Increments again

The chain rule idea can be motivated through increments.
Recall:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\Delta f}{\Delta x},$$

where Δf and Δx are increments,

$$\Delta f = f(x+h) - f(x), \qquad \Delta x = (x+h) - x.$$

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Of course the whole point is that differentiating f was hard, but differentiating g and k was easy.

We can introduce g and k by multiplying and dividing by another increment.

Intuition through increments

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\Delta f}{\Delta x}, \\
&= \lim_{h\to 0} \frac{g(k(x+h)) - g(k(x))}{h} \\
&= \lim_{h\to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h) - k(x)} \underbrace{\frac{k(x+h) - k(x)}{h}}_{k'(x) \text{ as } h \to 0}
\end{aligned}
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The second term would turn into the derivative of k.

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\lim_{h\to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h) - k(x)} \text{``}=\text{''} \lim_{h\to 0} \frac{\Delta g}{\Delta k} = g'(k(x))
$$

There is some new notation we haven't really seen before:

$$
g'(k(x)) \text{ means the function } g' \text{ evaluated at } k(x).
$$

The chain rule

Putting things together:

$$f'(x) = \lim_{h \to 0} \frac{g(k(x+h)) - g(k(x))}{k(x+h) - k(x)} \, \frac{k(x+h) - k(x)}{h} = g'(k(x))k'(x)$$

Informally, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{\Delta f}{\Delta x} = \frac{\Delta g(k(x))}{\Delta x} = \frac{\Delta g}{\Delta k} \, \frac{\Delta k}{\Delta x} \approx \frac{\mathrm{d}g}{\mathrm{d}k}\Big|_{k=k(x)} \frac{\mathrm{d}k}{\mathrm{d}x}$$

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Formally, we have motivated the chain rule:

Theorem (The Chain Rule)

Given differentiable functions $k(x)$ *and* $g(x)$*, suppose that* $f(x) = g(k(x))$. *Then:*

$$f'(x) = g'(k(x))\, k'(x)$$

This rule is nontrivial, but mastery allows you to differentiate almost any function you can write down.

Example

Compute the derivative of $f(x) = (x^2 + 3)^{45}$.
(Ans: $f'(x) = 90x(x^2 + 3)^{44}$.)

Example

Compute the derivative of $f(x) = \sin(x^2)$.
(Ans: $f'(x) = 2x\cos(x^2)$.)

Example

Compute the derivative of $f(x) = \tan^4 x$.
(Ans: $f'(x) = 4(\tan^3 x)(\sec^2 x)$.)

Example

Compute the derivative of $g(t) = \left(\frac{t^4+3t+2}{t^2+1}\right)^{12}$

(Ans: $g'(t) = 12\left(\frac{t^4+3t+2}{t^2+1}\right)^{11} \frac{(4t^3+3)(t^2+1)-2t(t^4+3t+2)}{(t^2+1)^2}$)

Example

Compute the derivative of $q(s) = \cos(\sin s^9)$
(Ans: $q'(s) = -9s^8(\sin(\sin s^9))(\cos s^9)$.)

The chain rule

The motivation for "chain rule" as a name can be understood as follows:

If $f(x) = a(b(c(x)))$, where a, b, and c are all differentiable, then,

$$f'(x) = a'(b(c(x))) \cdot b'(c(x)) \cdot c'(x),$$

and one can imagine how this might work for an arbitrary number of function compositions by chaining together individual derivatives.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.