Math 1210: Calculus I
Higher Order Derivatives

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.6

The derivative

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We can repeatedly take derivatives of f, resulting in higher order derivatives.

First, some terminology:

- We call $f'(x)$ the *first derivative* of f.
- $f'(x)$ is the *first-order derivative* of f.

Higher order derivatives

As one might expect, a second order derivative or second derivative for $f(x)$ is the derivative of the derivative of f:

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This process, including the prime notation, can be repeated indefinitely, in principle:

- f' is the first derivative of f
- f'' is the first derivative of f'. It's the second derivative of f.
- f''' is the first derivative of f''. It's the third derivative of f.
- f'''' is the first derivative of f'''. It's the fourth derivative of f.
- $\vdots$

A little practice

Example

Compute f', f'', f''', and f'''' when $f(x) = 4x^3 - 3x^2 + 8x - 5$
(Ans: $f'(x) = 12x^2 - 6x + 8$, $f''(x) = 24x - 6$, $f'''(x) = 24$, $f''''(x) = 0$)

$$f'(x) = 12x^2 - 6x + 8 + 0$$

$$f''(x) = \frac{d}{dx} f'(x) = \frac{d}{dx}(12x^2 - 6x + 8) = 24x - 6$$

$$f'''(x) = \frac{d}{dx}(24x - 6) = 24$$

$$f''''(x) = 0$$

More notation

Recall that we have some equivalent notations for the derivative:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = f^{(1)}(x)$$

$f^{(1)}$ ← 1st derivative

where the last notation is new.

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$\frac{d}{dx}\left(\frac{d}{dx}(f)\right)$

This extends to higher-order derivatives. E.g.,:

2nd derivative.

$$f''(x) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = f^{(2)}(x)$$

$$f'''(x) = \frac{\mathrm{d}^3}{\mathrm{d}x^3} f(x) = \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = f^{(3)}(x)$$

$f^2(x) \neq f^{(2)}(x)$

These alternative notations become convenient for very high order derivatives. (Imagine writing $f^{(13)}$ using prime notation.)

We don't write $\frac{df^2}{dx^2}$

More examples

Example

When $f(x) = \cos(3x)$, compute $\frac{d^4 f}{dx^4}$.

$f'(x) = -3 \sin 3x$ (chain rule)

$\frac{d^2 f}{dx^2} = -9 \cos(3x)$ (chain rule)

$f^{(3)}(x) = 27 \sin(3x)$ (chain rule)

$\frac{d^4 f}{dx^4} = 81 \cos(3x)$ (chain rule)

$f'(x) = -(3)^1 \sin 3x$

$f''(x) = -(3)^2 \cos 3x$

$f'''(x) = (3)^3 \sin 3x$

$f''''(x) = (3)^4 \cos 3x$

More examples

Example

When $f(x) = x^4 + x^3 + x^2$, compute $f^{(3)}(2)$.

$$f^{(1)}(x) = 4x^3 + 3x^2 + 2x$$

$$f^{(2)}(x) = 12x^2 + 6x + 2$$

$$f^{(3)}(x) = 24x + 6 \Longrightarrow f^{(3)}(2) = 48 + 6 = 54$$

Physical interpretation of higher order derivatives

If $f(t)$ represents the position of an object at time t, we've seen that $f'(t)$ is its instantaneous rate of change of position, i.e., velocity.

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- $f^{(3)}$ is the instantaneous change in acceleration. This is *jerk*.

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- $f^{(2)}$ is the instantaneous change in velocity. This is *acceleration*.
- $f^{(3)}$ is the instantaneous change in acceleration. This is *jerk*.
- $f^{(4)}$ is the instantaneous change in jerk. This is *snap*.
- $f^{(5)}$ is the instantaneous change in snap. This is *crackle*.
- $f^{(6)}$ is the instantaneous change in crackle. This is *pop*.

Velocity and acceleration example

Example (Example 3, Section 2.6)

An object moves along a horizontal coordinate line in such a way that its position at time t is specified by,

$$s(t) = t^3 - 12t^2 + 36t - 30$$

(Here s is measured in feet and t in seconds.)

1. When is the velocity 0?
2. When is the velocity positive?
3. When is the object moving to the left (that is, in the negative direction)?
4. When is the acceleration positive?

velocity $v(t) = s'(t) = 3t^2 - 24t + 36$

When is $v(t) = 0$?

Find t such that $3t^2 - 24t + 36 = 0$

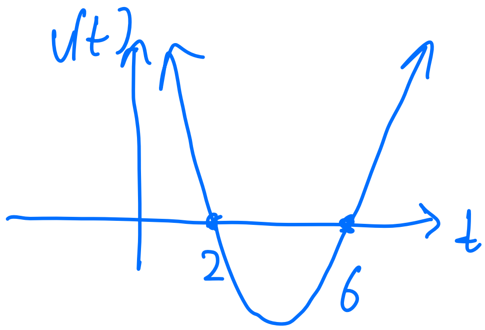
$t^2 - 8t + \cancel{9}\ 12 = 0$

$(t-2)(t-6) = 0$

(1): $t = 2, 6$

When is $v(t)$ positive?

$v(t) = 3t^2 - 24t + 36$



zeros at $t = 2, 6$

concave up

(2): $v(t) > 0$ when $t < 2$ and $t > 6$

When is the object moving to the left?

When is $v(t) < 0$?

From graph: (3) $v(t) < 0$ when $2 < t < 6$

When is the acceleration positive?

acceleration $a(t) = s''(t) = v'(t)$

$= 6t - 24$

When is $a(t) > 0$?

$6t - 24 > 0$

(4) $t > 4$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.