Math 1210: Calculus I
Implicit Differentiation

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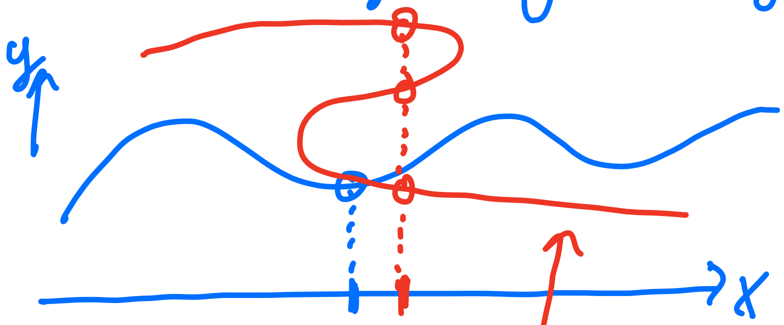
Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.7

Computing derivative

So far, we've focused on computing the derivative of a function $y(x)$, *assuming* that y is a given, explicit function of x.

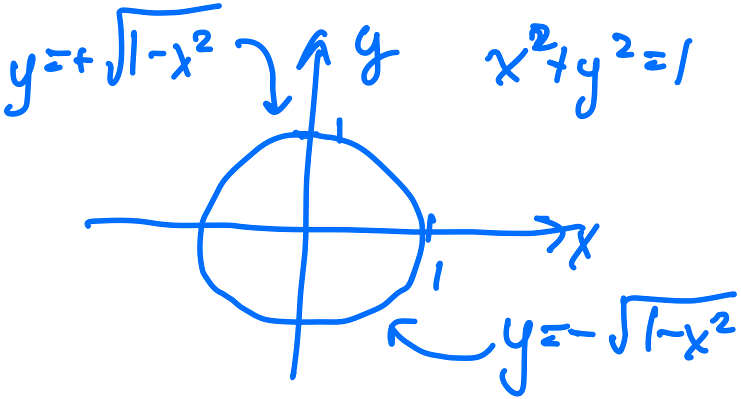
There are cases when y is an *implicitly* defined function of x.

when can I generally write $y = y(x)$?



Not a function

[But can still draw tangent lines)



Derivative of implicitly defined functions

When x and $y(x)$ are implicitly related, we can still compute y' using the chain rule.

Example

Compute $y'(x)$ if $y^4 + 5y = x^2$.
(Ans: $y'(x) = \frac{2x}{4y^3+5}$.)

Start: $y^4 + 5y = x^2$

Apply a derivative to both sides

"Apply d/dx"

RHS: $x^2 \xrightarrow{d/dx} 2x$

LHS: $y^4 + 5y \xrightarrow{d/dx} \frac{d}{dx}(y^4 + 5y) = \underbrace{\frac{d}{dx} y^4}_{??} + 5 \cdot \overbrace{\frac{d}{dx} y}^{y'(x)}$

Write $y^4(x)$ as a composition: $h(x) = x^4 \longrightarrow h'(x) = 4x^3$

$$y^4(x) = h(y(x))$$

$$\frac{d}{dx} y^4(x) = \frac{d}{dx} h(y(x))$$

$$= h'(y(x)) \cdot y'(x)$$

$$= 4\, y^3(x) \cdot y'(x)$$

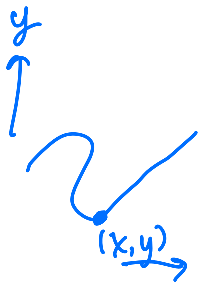
$$\frac{d}{dx}(y^4 + 5y) = 4y^3 \cdot y' + 5y'$$

LHS = RHS

$$4y^3 \cdot y' + 5y' = 2x$$

$$y'(x)[4y^3 + 5] = 2x$$

$$y'(x) = \frac{2x}{4y^3 + 5}$$



(It's ok if y' is expressed in terms of x and y)

Caveat: to evaluate y' at some value of x, we also need a corresponding value of y

The procedure from the previous example is called **implicit differentiation**. There are some things to observe:

- In general, as in the previous example, this produces a derivative y' that is expressed as a function of both x and y. (This is perfectly ok!)
- So long as y is differentiable, a derivative computed with implicit differentiation is the same as one computed if we had an explicit form for $y(x)$.
- Evaluating an implicitly computed derivative requires values for the pair (x, y); the point x alone is in general not enough.

Cf: last example: $y' = \frac{2x}{4y^3+5}$, $y^4 + 5y = x^2$

Can't "evaluate y' at $x=3$". (Also need y associated to $x=3$)

The circle

$y = x^3$

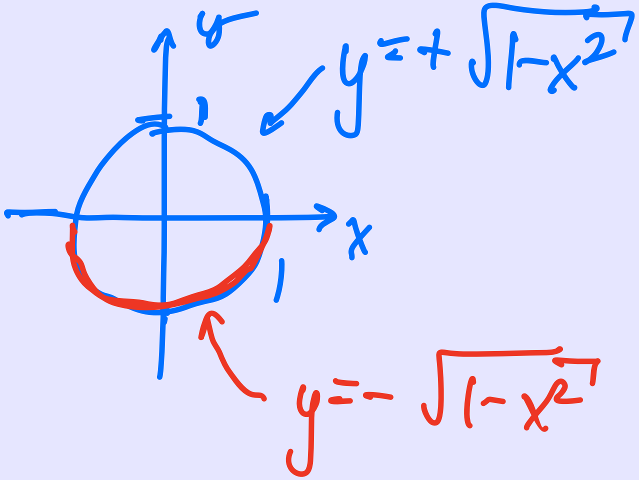
$\frac{dy}{dx} = 3x^2$

$y^2 = y(x) \cdot y(x)$

$\frac{d}{dx}\left(y(x) \cdot y(x) \right) = y' \cdot y + y \cdot y'$

Example

Consider the graph of the relation $x^2 + y^2 \overset{=}{\cancel{+}} 1$. Compute the slope of the tangent line to this graph using both implicit differentiation, and through explicit means.



Implicit differentiation:

$\frac{d}{dx}\left(x^2 + y^2 = 1 \right)$

$2x + 2y \cdot y' = 0$ (chain rule)

$y' = \frac{-x}{y}$

"Explicit" derivative: upper half of graph

$$y = \sqrt{1-x^2}$$
$$= (1-x^2)^{1/2}$$

fact: power rule works for fractional powers

$y(x) = h(g(x))$ $\quad g(x) = 1-x^2 \rightarrow g'(x) = -2x$

$\quad h(x) = x^{1/2} \longrightarrow h'(x) = \frac{1}{2}x^{-1/2}$

$$\Rightarrow y'(x) = h'(g(x)) \cdot g'(x)$$
$$= \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$$
$$= \frac{-x}{\sqrt{1-x^2}}$$

Two formulas: $y' = \frac{-x}{y}$, $y' = \frac{-x}{\sqrt{1-x^2}}$

Since $y = \sqrt{1-x^2}$, these are the same.

Exercise: verify also for lower half, $y = -\sqrt{1-x^2}$

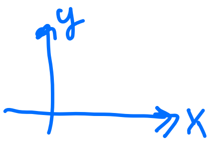
Aside:

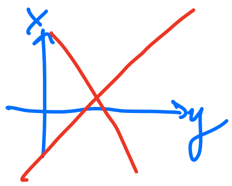
$x^2+y^2=1$

why not take d/dy ?

d/dx: operation to compute slope of tangent line to a graph.

d/dy: operation to compute slope of tangent line to a graph of x as a function of y





If we did take d/dy:

$$x^2+y^2=1$$

$$\Big\downarrow d/dy$$

$$2x\cdot\frac{dx}{dy}+2y=0$$

Example (Example 2.7.1)

Find $y'(x)$ if $4x^2y - 3y = x^3 - 1$

(Ans: $y'(x) = \frac{3x^2-8xy}{4x^2-3}$, but could also be an explicit function of x.)

d/dx on both sides: $\frac{d}{dx}(4x^2y) - 3y' = 3x^2$

$$\frac{d}{dx}(4x^2y) = 4\frac{d}{dx}(x^2 \cdot y) = 4(2x \cdot y + x^2 \cdot y') \quad \text{(product rule)}$$

$$= 8xy + 4x^2y'$$

$$\to 8xy + 4x^2y' - 3y' = 3x^2 \longrightarrow y' = \frac{3x^2 - 8xy}{4x^2-3}$$

Example (Example 2.7.3)

Compute the equation of the tangent line to the curve $y^3 - xy^2 + \cos(xy) = 2$ at the point $(0, 1)$.
(Ans: $y = x/3 + 1$)

slope @ $(0,1)$

d/dx of $y^3 - xy^2 + \cos(xy) = 2$

$3y^2 \cdot y' - [1 \cdot y^2 + x \cdot 2yy'] - \sin(xy) \cdot \frac{d}{dx}(xy) = 0$

$3y^2 \cdot y' - [y^2 + 2xyy'] - \sin(x \cdot y)[1 \cdot y + x \cdot y'] = 0$

(continue next frame)

What is $y'(x)$ at $(x,y)=(0,1)$?

$$\underset{\uparrow\, y=1}{3y^2y'} - [\underset{\uparrow\, y=1}{y^2} + 2\cancel{x}^{\,x=0}yy'] - \sin(\cancel{x}^{\,x=0}y)\,[y+xy'] = 0$$

$$3\cdot y' - 1 = 0 \rightarrow y' = 1/3 \quad (\text{at } (x,y)=(0,1))$$

Equation of line through $(0,1)$, slope $1/3$.

$$y-1 = \tfrac{1}{3}(x-0)$$

$$\boxed{y = \frac{x}{3} + 1}$$

The "fractional" power rule

Recall that $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ if n is any integer.

What if n is a *rational* number? I.e., suppose $n = p/q$ for integers p, q. $(q \neq 0)$

The "fractional" power rule

Recall that $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ if n is any integer.

What if n is a *rational* number? I.e., suppose $n = p/q$ for integers p, q.

Theorem

If n is any rational number, then $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$.

Why? $y(x) = x^n = x^{(p/q)} \Longleftrightarrow y^q = x^p$

$\downarrow$ implicit differentiation

$y'(x) = (p/q)\, x^{(p/q - 1)}$

Example

Compute $y'(x)$ if $x^{5/3} + y^{13/4} = \sqrt{x^2+1}$.
(Ans: $y' = \frac{4}{13} y^{-9/4} \left(\frac{x}{\sqrt{x^2+1}} - \frac{5}{3} x^{2/3} \right)$.)

$y^{13/4} = h(y(x))$, $h(x) = x^{13/4}$

$$x^{5/3} + y^{13/4} = (x^2+1)^{1/2}$$

$$\frac{5}{3} x^{2/3} + \frac{13}{4} y^{9/4} \cdot y' = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x$$ (chain rule twice)

$$y' = \frac{x(x^2+1)^{-1/2} - \frac{5}{3} x^{2/3}}{\frac{13}{4} y^{9/4}}$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.