Math 1210: Calculus I
Implicit Differentiation

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.7

Computing derivative

So far, we've focused on computing the derivative of a function $y(x)$, *assuming* that y is a given, explicit function of x.

There are cases when y is an *implicitly* defined function of x.

When x and $y(x)$ are implicitly related, we can still compute y' using the chain rule.

Example

Compute $y'(x)$ if $y^4 + 5y = x^2$.
(Ans: $y'(x) = \frac{2x}{4y^3+5}$.)

Implicit differentiation

The procedure from the previous example is called **implicit differentiation**. There are some things to observe:

- In general, as in the previous example, this produces a derivative y' that is expressed as a function of <u>both</u> x and y. (This is perfectly ok!)
- So long as y is differentiable, a derivative computed with implicit differentiation is the same as one computed if we had an explicit form for $y(x)$.
- Evaluating an implicitly computed derivative requires values for the pair (x, y); the point x alone is in general not enough.

Example

Consider the graph of the relation $x^2 + y^2 + 1$. Compute the slope of the tangent line to this graph using both implicit differentiation, and through explicit means.

Examples

Example (Example 2.7.1)

Find $y'(x)$ if $4x^2y - 3y = x^3 - 1$

(Ans: $y'(x) = \frac{3x^2 - 8xy}{4x^2 - 3}$, but could also be an explicit function of x.)

Example (Example 2.7.3)

Compute the equation of the tangent line to the curve $y^3 - xy^2 + \cos(xy) = 2$ at the point $(0, 1)$.
(Ans: $y = x/3 + 1$)

The "fractional" power rule

Recall that $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$ if n is any integer.

What if n is a *rational* number? I.e., suppose $n = p/q$ for integers p, q.

The "fractional" power rule

Recall that $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ if n is any integer.

What if n is a *rational* number? I.e., suppose $n = p/q$ for integers p, q.

Theorem

If n is any rational number, then $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$.

Example

Example

Compute $y'(x)$ if $x^{5/3} + y^{13/4} = \sqrt{x^2 + 1}$.
(Ans: $y' = \frac{4}{13} y^{-9/4} \left(\frac{x}{\sqrt{x^2+1}} - \frac{5}{3} x^{2/3} \right)$.)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.