Math 1210: Calculus I
Related Rates

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.8

What are related rates?

In real-world modeling, we often encounter situations where *rates of change* of two variables are related to one another.

For conceptual simplicity:

- Let t be time (in some units)
- Let $x(t)$ be a time-dependent quantity (say, a position or distance)
- Let $y(t)$ be another time-dependent quantity (say, another position or distance)

What are related rates?

In real-world modeling, we often encounter situations where *rates of change* of two variables are related to one another.

For conceptual simplicity:

- Let t be time (in some units)
- Let $x(t)$ be a time-dependent quantity (say, a position or distance)
- Let $y(t)$ be another time-dependent quantity (say, another position or distance)
- Suppose x and y are quantities that are related to each other through some equation (the area of a circle is related to is radius).
- Then $\frac{\mathrm{d}}{\mathrm{d}t}x$ and $\frac{\mathrm{d}}{\mathrm{d}t}y$ are related to each other through implicit differentiation.
- This allows us to compute $\frac{\mathrm{d}}{\mathrm{d}t}x$ if we know $\frac{\mathrm{d}}{\mathrm{d}t}y$ at some time.

What are related rates?

In real-world modeling, we often encounter situations where *rates of change* of two variables are related to one another.

For conceptual simplicity:

- Let t be time (in some units)
- Let $x(t)$ be a time-dependent quantity (say, a position or distance)
- Let $y(t)$ be another time-dependent quantity (say, another position or distance)
- Suppose x and y are quantities that are related to each other through some equation (the area of a circle is related to is radius).
- Then $\frac{\mathrm{d}}{\mathrm{d}t}x$ and $\frac{\mathrm{d}}{\mathrm{d}t}y$ are related to each other through implicit differentiation.
- This allows us to compute $\frac{\mathrm{d}}{\mathrm{d}t}x$ if we know $\frac{\mathrm{d}}{\mathrm{d}t}y$ at some time.

In terms of mechanics, these types of problems are exercises in implicit differentiation.

The most difficult part of these problems is typically deriving the equation on which to exercise differentiation.

Simpler examples

Example

Suppose a spherical balloon is being inflated at a rate of $3\mathrm{cm}^3$ per second. How fast is the radius increasing when the balloon has volume $100\mathrm{cm}^3$?

$$V = \frac{4\pi}{3} r^3$$

$\underbrace{\text{rate of increase of } r}_{\frac{dr}{dt}}$ when V is $100\mathrm{cm}^3$?

Things change in time: $V = V(t)$, $r = r(t)$ $\longrightarrow$ $V(t) = \frac{4}{3}\pi r^3(t)$

$$\xrightarrow{d/dt} V'(t) = \frac{4\pi}{3} 3 r^2(t) \cdot r'(t)$$

$$r'(t) = \frac{V'(t)}{4\pi r^2(t)}$$

want: $r'(t)$ when V is $100\,cm^3$

when V is $100\,cm^3$, what is $V'(t)$ and $r(t)$?

Problem statement: $\frac{dV}{dt} = 3\,cm^3/s$

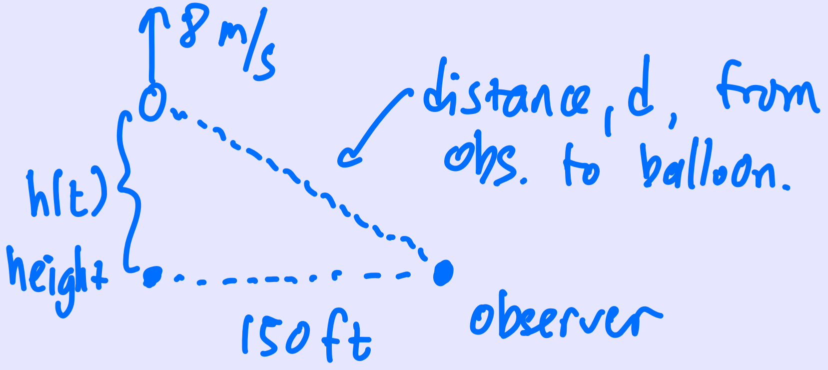
$$V = \frac{4\pi}{3}r^3 \xrightarrow{V=100} 100 = \frac{4\pi}{3}r^3$$

$$r = \left(\frac{75}{\pi}\right)^{1/3}$$

$$\Longrightarrow r'(t) = \frac{V'(t)}{4\pi r^2(t)} \underset{\text{when } V=100}{=} \frac{3}{4\pi\left(\frac{75}{\pi}\right)^{2/3}} \underset{\text{if have a calc.}}{=} \text{(write as a decimal)}$$

Example (Example 2.8.1)

A small balloon is released at a point 150 feet away from an observer, who is on level ground. If the balloon goes straight up at a rate of 8 feet per second, how fast is the distance from the observer to the balloon increasing when the balloon is 50 feet high?



Q: what is $d'(t)$ when $h(t) = 50$?

$$d^2(t) = h^2(t) + (150)^2$$

when $h(t) = 50$, $d^2(t) = 50^2 + (150)^2$

$$= 50^2 \cdot 10$$

$$= 25{,}000$$

$$d(t) = 50\sqrt{10}$$

$$d^2(t) = h^2(t) + (150)^2$$

$$\xrightarrow[\text{wrt}/t]{\text{deriv.}} \quad 2 \cdot d(t) \cdot d'(t) = 2h(t)h'(t)$$

$$d'(t) = \frac{h(t) \cdot h'(t)}{d(t)}$$

Q: What is $d'(t)$ when $h(t) = 50$?

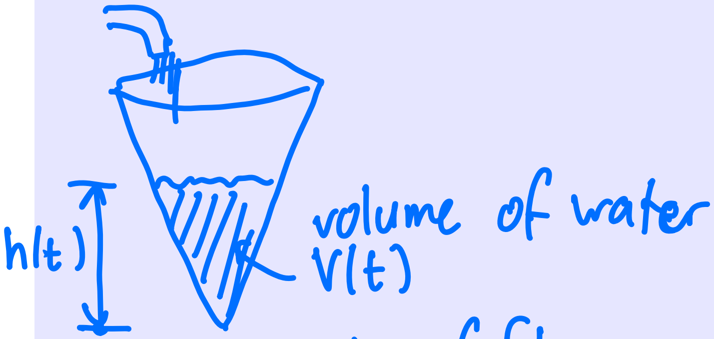
At this time, $d(t) = 50\sqrt{10}$.

Problem statement: $h'(t) = 8$

$$d'(t) = \frac{h(t) \cdot h'(t)}{d(t)} = \frac{50 \cdot 8}{50\sqrt{10}} = \frac{8}{\sqrt{10}} \left[ft/s \right]$$

Example (Example 2.8.2)

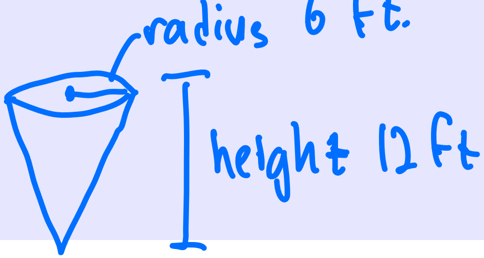
Water is pouring into a conical tank at the rate of 8 cubic feet per minute. If the height of the tank is 12 feet and radius of its circular opening is 6 feet, how fast is the water level rising when the water is 4 feet deep?



$$\frac{dV}{dt} = 8 \frac{ft^3}{s}$$

$h(t)$ = water height inside tank

Q: What is dh/dt when $h(t) = 4$?

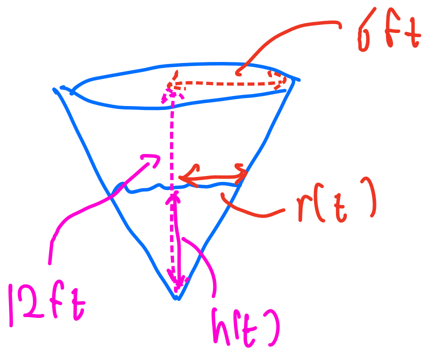


Volume of a cone with height h and circular radius r is $\frac{\pi}{3} h r^2$

If $r(t)$ is the radius of the cone corresponding to water level, then $V(t) = \frac{\pi}{3} h(t) r^2(t)$

Volume of entire tank is $\frac{\pi}{3} r^2 h = \frac{\pi}{3} 6^2 \cdot 12$

$= 144\pi \, [ft^3]$



The triangle formed by $h(t)$ and $r(t)$ is <u>similar</u> to the triangle formed by 12ft and 6ft.

$\Longrightarrow \frac{r(t)}{6} = \frac{h(t)}{12}$ (ratios of lengths are the same).

$\Longrightarrow r(t) = h(t)/2$

$\Longrightarrow V(t) = \frac{\pi}{3} h(t) r^2(t)$

$= \frac{\pi}{12} h^3(t)$

$\xrightarrow{\text{take } d/dt} V'(t) = \frac{\pi}{4} h^2(t) \cdot h'(t)$

recall: $dV/dt = 8$, choose t s.t. $h(t) = 4$.

$\Longrightarrow h'(t) = \frac{4V'(t)}{\pi h^2(t)} = \frac{4 \cdot 8}{\pi \cdot 16} = \frac{2}{\pi}$ ft/~~s~~min

Related rates problems

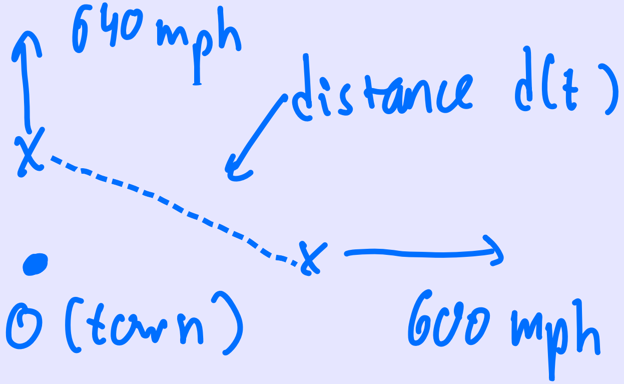
There are some common steps for related rates problems:

- Typically drawing a diagram first to visualize the problem is helpful.
- The diagram typically can be used to form an equation that relates various quantities in the diagram.
- Implicit differentiation of this equation yields an expression for the desired rate (or derivative)
- Some additional work may be needed to determine values of all the quantities in the differentiated expression.

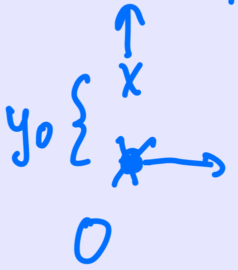
(The book has a more wordy description of these steps.)

Example (Example 2.8.3)

An airplane flying north at 640 miles per hour passes over a certain town at noon. A second airplane going east at 600 miles per hour is directly over the same town 15 minutes later. If the airplanes are flying at the same altitude, how fast will they be separating at 1:15pm?



At 12:15pm:

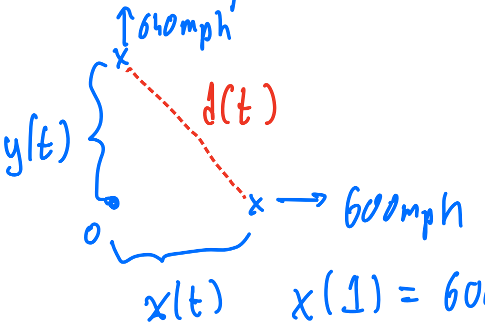


y_0: distance traveled by airplane at 640 mph for 15 min

$\Rightarrow y_0 = \frac{640}{4} = 160$ miles

At 1:15pm:

$y(1) = 640 \cdot 1 + y_0 = 640 + 160 = 800$ miles



$x(1) = 600 \cdot 1 = 600$ miles

$$x^2(t) + y^2(t) = d^2(t)$$

$$\xrightarrow{d/dt} 2x(t)\cdot x'(t) + 2y(t)\,y'(t) = 2d(t)\cdot d'(t)$$

evaluate @ $t=1$:

want this

$$x(1)\cdot x'(1) + y(1)\cdot y'(t) = d(1)\cdot d'(1)$$

600

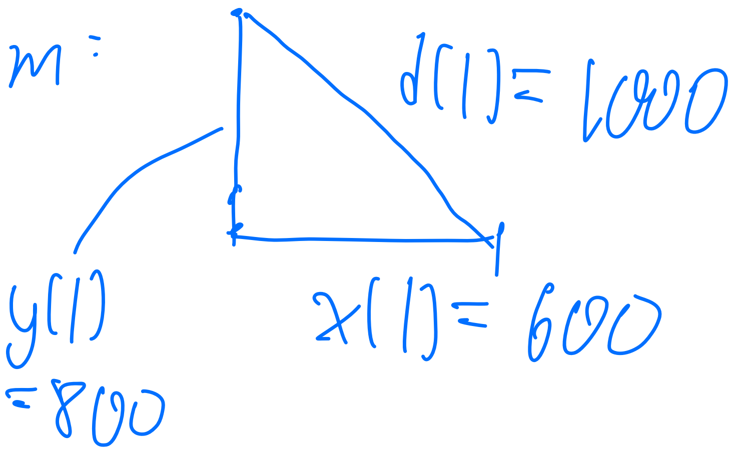
speed of airplane ~~1~~ 2, ~~640~~ 600

800

speed of airplane 1, 640

distance between planes at 1:15pm

At 1:15pm:



$$\Rightarrow d'(1) = \frac{x(1)x'(1) + y(1)y'(1)}{d(1)}$$

$$= \frac{600 \cdot 600 + 800 \cdot 640}{1000}$$

$$= 60 \cdot 6 + 8 \cdot 64$$

$$= 360 + \overset{512}{\cancel{480}} = \underline{\overset{872}{\cancel{840}} \text{ mph}}$$

More examples

Example (Example 2.8.4)

A woman standing on a cliff is watching a motorboat through a telescope as the boat approaches the shoreline directly below her. If the telescope is 250 feet above the water level and if the boat is approaching at 20 feet per second, at what rate is the angle of the telescope changing when the boat is 250 feet from the shore?

Example (Example 2.8.5)

As the sun sets behind a 120-foot building, the building's shadow grows. How fast is the shadow growing (in feet per second) when the sun's rays make an angle of 45 degrees (or $\pi/4$ radians)?

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.