Math 1210: Calculus I
Approximations and differentials

Department of Mathematics, University of Utah

Spring 2025

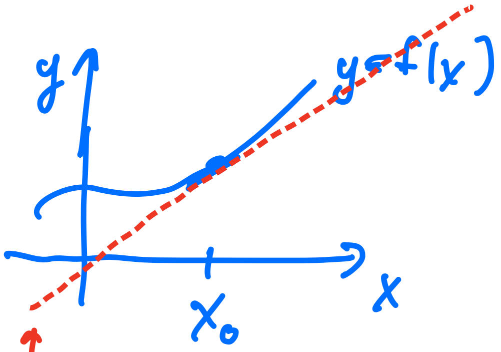
Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 2.9

Derivatives and approximations

One of the most helpful uses of derivatives is in approximations. Recall the definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

One of our interpretations of $f'(x_0)$ is that it is the slope of the tangent line to the graph of f at the value $x = x_0$.



"Close" to x_0, then $y=f(x)$ and $y=L(x)$ look very similar

$y=L(x)$, the equation of the tangent line to f at x_0.

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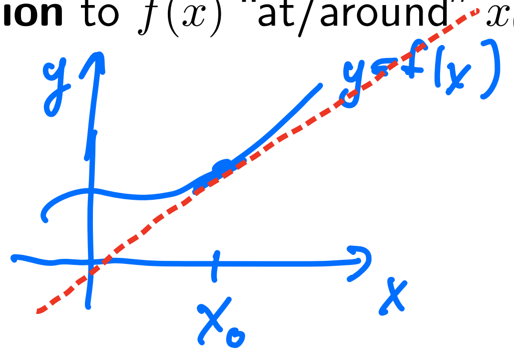
Tangent lines, around $x = x_0$, are "close" to the function f, and so in lieu of evaluating $f(x)$, we could (a) form a tangent line at $x = x_0$, and (b) evaluate the tangent line value at any $x \neq x_0$.

We could call the tangent line at $x = x_0$ a **linear approximation** to $f(x)$ "at/around" x_0.

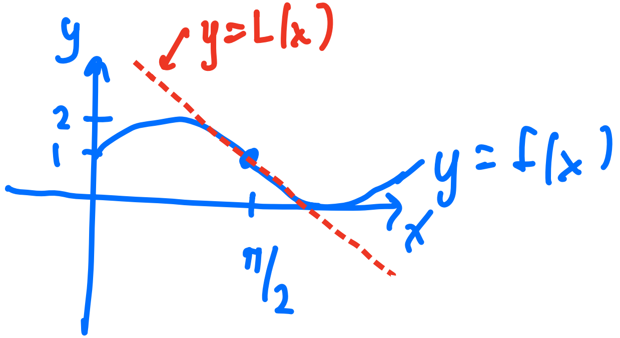
Conceptually, one expects that this linear approximation

- is a good approximation to $f(x)$ when x is close to x_0
- becomes a poorer approximation when x is far from x_0



Example (Based on example 2.9.6)

Consider $f(x) = 1 + \sin(2x)$. Compute the linear approximation $L(x)$ to f at $x = \pi/2$. Use the linear approximation to estimate $f(1.5)$.



$x = \pi/2 \Rightarrow f(x) = 1 + \sin\pi = 1$

$f'(x) = 2\cos(2x)$

$f'(\pi/2) = 2\cos(\pi) = -2$ (slope of $L(x)$)

point on $L(x)$: $(\pi/2, 1)$

$y - 1 = -2(x - \pi/2) \rightarrow y = -2x + \pi + 1 = L(x)$

@ $x = 1.5$: $y = -2(1.5) + \pi + 1 = \pi - 2 \approx f(1.5)$

Approximations with increments

Linear approximations are a specific strategy to use the derivative to approximate the function.

There is another way that can be motivated from the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

The idea: if we have access to $f'(x)$, we can *approximate* the change in f:

$$f'(x) \approx \frac{\Delta f}{\Delta x} \text{ for small } \Delta x \Longrightarrow \Delta f \approx f'(x) \Delta x \text{ for small } \Delta x.$$

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$$f'(x) \approx \frac{\Delta f}{\Delta x} \text{ for small } \Delta x \Longrightarrow \Delta f \approx f'(x)\Delta x \text{ for small } \Delta x.$$

Writing this another way: suppose we know $f'(x)$ and $f(x)$, and we are given some x perturbation Δx. Then:

$$\Delta f \approx f'(x)\Delta x \quad \Longrightarrow \quad f(x + \Delta x) \approx f(x) + f'(x)\Delta x$$

This is yet another way to compute approximations.

Example: approximations with increments

Example (Based on example 2.9.6)

Consider $f(x) = 1 + \sin(2x)$. Use increments to estimate $f(1.5)$ from $f(x)$ and $f'(x)$ at $x = \pi/2$.

$\Delta f \approx f'(x)\,\Delta x$

$f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$

$x = \pi/2$

want $x + \Delta x = 1.5$

$\Delta x = 3/2 - \pi/2$

at $x = \pi/2$: $f(\pi/2) = 1$, $f'(\pi/2) = -2$

$f(1.5) = f(x+\Delta x) \approx 1 + (-2)\left(\frac{3}{2} - \pi/2\right) = \pi -$

Tangent line and increment approximations are the same

From the previous two examples: the same answer was achieved in two different ways.

This can be seen by first computing the linear approximation $L(x)$ to f at the point x_0:

$$\text{Line through } (x_0, f(x_0)) \text{ with slope } f'(x_0)\text{: } y - f(x_0) = f'(x_0)(x - x_0)$$

Hence, the line approximation is $f(x) \approx L(x) = f(x_0) + f'(x_0)(x - x_0)$.

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If we write the increment $\Delta x = x - x_0$, we have:

$$\Delta f = f(x) - f(x_0) \approx L(x) - f(x_0) = f'(x_0)(x - x_0) = f'(x_0)\Delta x.$$

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Therefore: one could approximate $f(x)$ with x close to a value x_0 by either,

- Compute $L(x)$, which is the value of the tangent line to f at x_0
- Compute $\Delta f \approx f'(x_0)\Delta x$.

These are both equivalent.

Differentials: Notation for approximations with increments

For the purposes of approximation, it's useful to introduce extra notation.

Let $f(x)$ be a function, and let x_0 be a fixed value of x. Then:

- $\Delta x = x - x_0$ is an *increment* of x
- $\Delta f = f(x) - f(x_0)$ is an *increment* of f

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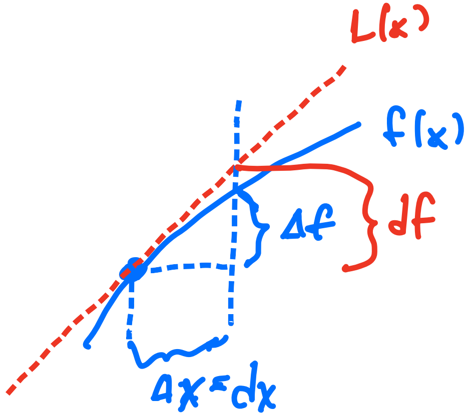
- $\Delta f = f(x) - f(x_0)$ is an *increment* of f
- $\mathrm{d}x = x - x_0$ is a **differential** of x. It's equal to Δx
- $\mathrm{d}f = f'(x_0)(x - x_0)$ is a **differential** of f. It's not Δf.

The quantities Δx and Δf are exact increments of x and f.



The quantities $\mathrm{d}x$ and $\mathrm{d}f$ are differentials, and $\mathrm{d}f$ is an approximation to Δx.

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Differential notation is somewhat easy to remember, because it can be recovered by "multiplying" by $\mathrm{d}x$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = f'(x) \qquad \mathrm{d}f = f'(x)\mathrm{d}x.$$

Differential examples

Example (Example 2.9.1)

Compute $\mathrm{d}y$ if

1. $y = x^3 - 3x + 1$
2. $y = \sqrt{x^2 + 3x}$
3. $y = \sin(x^4 - 3x^2 + 11)$

1.) $y'(x) = 3x^2 - 3$

$dy = (3x^2 - 3)\,dx$

2.) $y'(x) = \frac{1}{2}(x^2+3x)^{-1/2} \cdot (2x+3)$

$dy = (x + 3/2)(x^2+3x)^{-1/2}\,dx$

Example (≈Example 2.9.2)

Compute approximations to $\sqrt{4.3}$ and $\sqrt{8.8}$ without a calculator.

Idea: $f(x) = \sqrt{x}$, at $x = 4$, $\Delta x = 0.3$, $df \simeq \Delta f$

$f(4) = 2$ $\quad f(x) = x^{1/2}$ (power rule)

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \rightarrow f'(4) = \frac{1}{2} \cdot \frac{1}{2} = 1/4$

$df = f'(x)\,dx = \frac{1}{4} \cdot 0.3 = 3/40$ (at $x = 4$, $dx = 0.3$)

$f(4 + \Delta x) \simeq f(4) + df = 2 + 3/40 = 2.075$

Example

The radius of a solid sphere is measured as 1m with an error of ± 0.01m. Use differentials to estimate the error in the volume caused by this error in the radius.

$$V(r) = \frac{4}{3}\pi r^3$$

$$dV = V'(r)\,dr$$

$$= 4\pi r^2\,dr$$

$$dr = \pm 0.01, \quad r = 1\text{m} \implies dV = \pm 4\pi (1)^2 \cdot 0.01$$

$$= \pm \frac{\pi}{25} \; [m^3]$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.