Math 1210: Calculus I
Maxima and minima

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.1

Optimization

A fairly common practical goal is to *optimize* something:

- Minimize the cost of transporting packages to consumers
- Minimize the drag over an airfoil
- Maximize the strength of a bridge
- Minimize the risk of losing money in financial markets
- Maximize the top velocity or acceleration of a car
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In all these cases, the "thing" being optimized (the objective) is a function of the state or properties of the system.

- Drag is a function of the shape of an airfoil
- Bridge strength is a function of bridge shape and geometry
- Time spent in traffic is a function of route chosen and time of day
- $\vdots$

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Calculus gives us fundamental tools for solving optimization problems.

One-dimensional functions

We'll mostly consider a very simple example of optimization in this course:

- Let f be an objective
- Let f be a function of the state, x.
- The domain of x is S, typically an interval on the real line.
- We seek to extrema (maxima or minima) of f over S.

We need to formalize maxima and minima.

compute/identify

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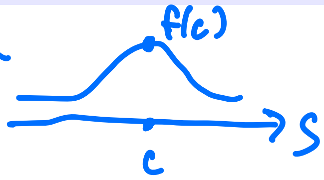
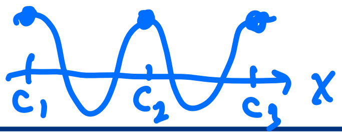
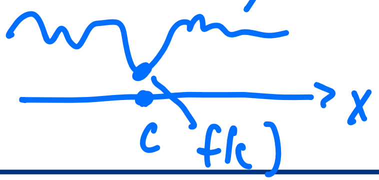
Definition

Let $f(x)$ be a function, where x is in some domain S. Let c be some point in S. Then:

- $f(c)$ is the **maximum value** of f over S if $f(c) \geqslant f(x)$ for all x in S.
- $f(c)$ is the **minimum value** of f over S if $f(c) \leqslant f(x)$ for all x in S.
- $f(c)$ is an **extreme value** if it's either a maximum or a minimum.

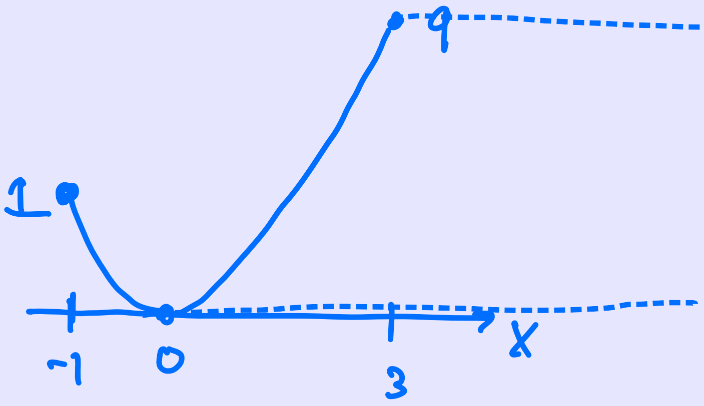






Example

Identify the maximum and minimum values of $f(x) = x^2$ over $[-1, 3]$.
(Ans: max value 9 at $x = 3$, min value 0 at $x = 0$.)



max value 9 (@x=3)

min value 0 (@ x=0)

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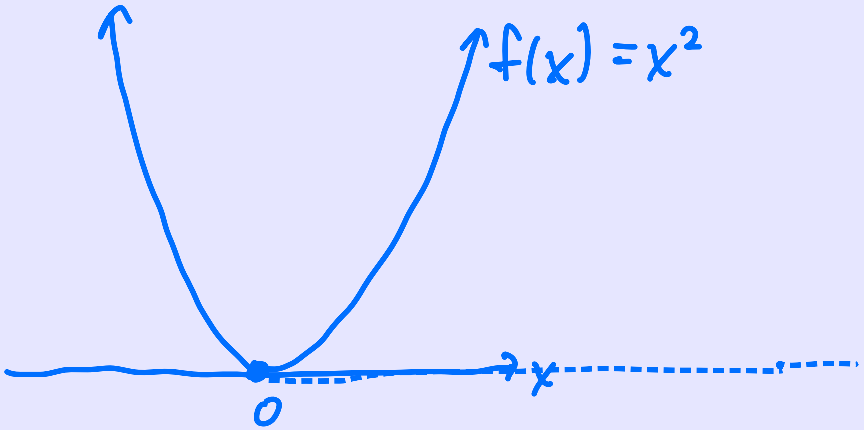
Identify the maximum and minimum values of $f(x) = x^2$ over $(-\infty, \infty)$.
(Ans: min value 0 at $x = 0$, no max value.)



There is no maximum
(no real number is the max value attained by f)

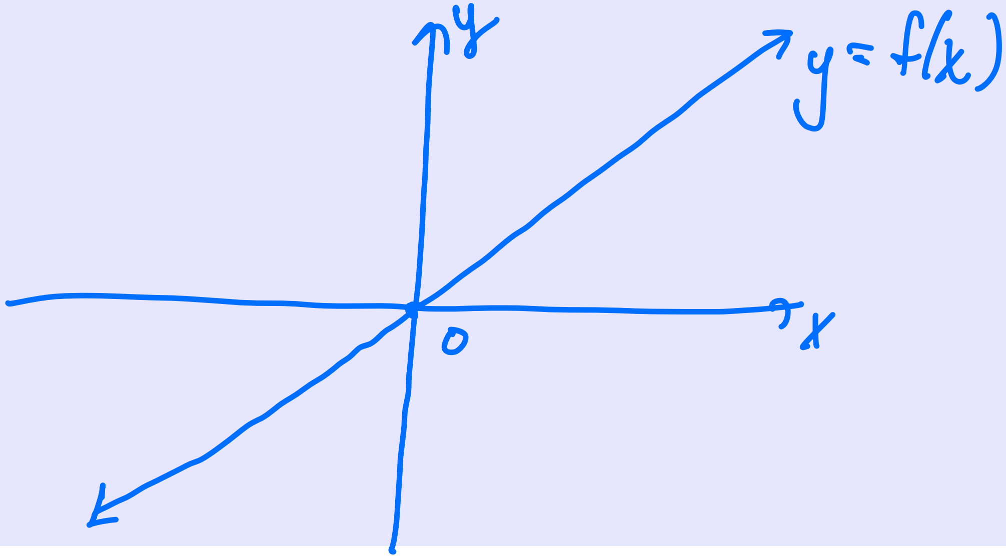
min value of 0 (@ $x=0$)

NB: The domain ("S") matters!

Example

Identify the extreme values of $f(x) = x$ over $(-\infty, \infty)$.
(Ans: no max value, no min value.)

no minimum, no maximum



Existence

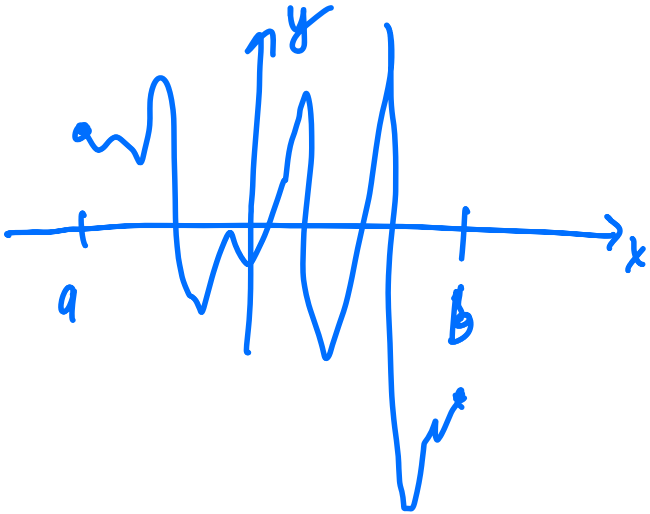
The previous examples raise an annoying point: it's possible that extreme values don't exist.

However, there is a fairly transparent set of assumptions that ensure existence.

Theorem

Suppose f is a continuous function over the interval $[a, b]$. Then f has a maximum and minimum value on this domain.

"Why?"



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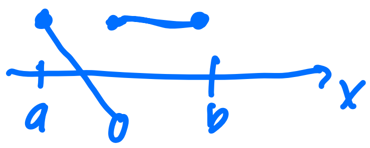
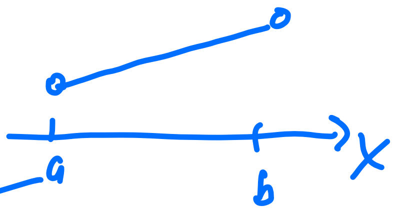
Key points:

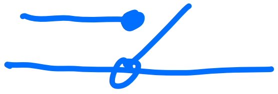
- f must be continuous
- The domain must be a closed interval

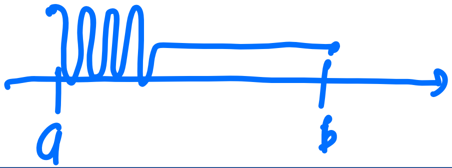




This theorem says nothing about where these extreme values occur. There could be numerous x locations that achieve the minimum or maximum value of f.
(E.g., $f(x) = \sin x$ for x in $[-3\pi, 3\pi]$.)

$$f(x) = \begin{cases} x, & x > 0 \\ 1, & x \leq 0 \end{cases}$$





$f(x) = 1$ on an(y) interval: min value $\underline{1}$

max value $\underline{1}$

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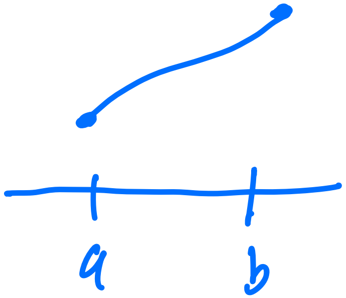
If the conditions of the theorem are violated, then in general a function could have just a minimum value, or just a maximum value, or both, or neither.

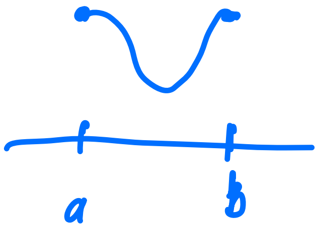
Where do extreme values occur?

A crucial point for us is to understand how to identify values of c for which $f(c)$ is an extreme value of f.

There are three sets of candidates for c:

– When the domain is a closed interval, i.e., $[a, b]$, the endpoints of this domain ($c = a$ and $c = b$)





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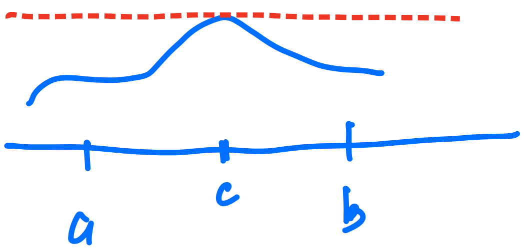
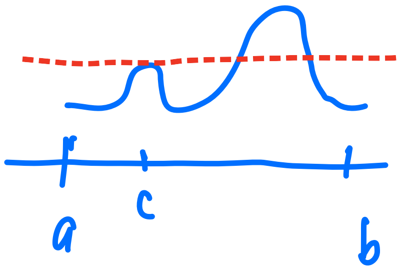
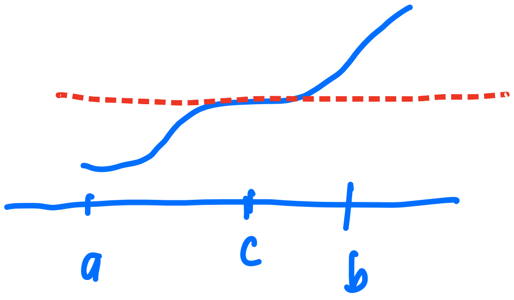
- **Stationary points**, i.e., points c where $f'(c) = 0$

$f'(c) = 0 \implies$ tangent line @ $x = c$ is horizontal







(e.g. $f(x) = x^3$)

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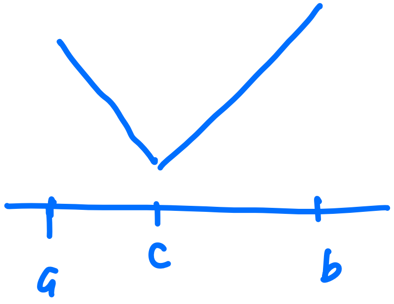
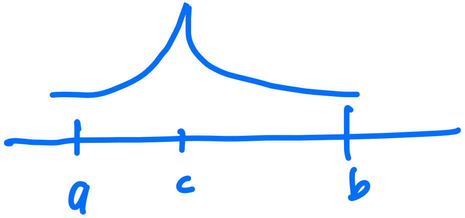
- **Stationary points**, i.e., points c where $f'(c) = 0$
- **Singular points**, i.e., points c where $f'(c)$ doesn't exist

E.g.: $f(x) = |x|$ @ $x = 0$

$f(x) = x^{2/3}$ @ $x = 0$





$f'(c)$ DNE : c is a singular point (cusps, corners)

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The set of **critical points** of f over some domain, is the collection of its endpoints (when the domain is closed) and its stationary points and its singular points.

Critical points are reasonable candidates for points corresponding to extreme values of f.

Critical points are sufficient

In fact, critical points are the only candidates for extreme values.

Theorem

Let f be a function on some interval I (not necessarily closed). Suppose there is some value *c in I such that $f(c)$ is an extreme value of f. Then $x = c$ is a critical point of f.*

$x=c$ is a critical point $\not\Longrightarrow$ $f(c)$ is an extreme value of f.

E.g. $f(x) = x^3$

$x=0$ is a critical point (stationary point).

$f(0)=0$ is neither a min. nor max.

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We also know that if I is a closed interval and f is continuous, then both minimum and maximum values of f exist, and hence in this special case we can always find extreme values by identifying critical points.

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The *converse* of this theorem is <u>not</u> true! If c is a critical point, it need not correspond to an extreme value!

An algorithmic procedure

Let f be a continuous function on a closed interval $[a, b]$. Then we can:

1. Identify the critical points of f on $[a, b]$. (I.e., the endpoints, stationary points, and singular points)

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What if I is not closed, or f is not continuous?
The above procedure can still be used, but step 2 is not necessarily correct. (We don't know that extreme values exist.)

Ex. $f(x) = x^2$

Compute maximum of f on $I = (-\infty, \infty)$

Compute critical points: no endpoints
stationary points: $f'(x) = 0 \Rightarrow 2x = 0$, $x = 0$
singular points: $f'(x)$ exists everywhere

max value of f on I is max value of f over critical points

WRONG

$\Longrightarrow f(x=0)=0 \Longrightarrow 0$ is the max value of f.

Example (Example 3.1.3)

Find the maximum and minimum values of $f(x) = -2x^3 + 3x^2$ on $[-\frac{1}{2}, 2]$. Also identify the x locations corresponding to these extreme values.
(Ans: Min value -4 at $x = 2$, max value 1 at $x = -1/2, 1$.)

$I = [-1/2, 2]$ is closed.
f is continuous

compute critical points.

endpoints: $x = -1/2$, $x = 2$

stationary points:
$f'(x) = -6x^2 + 6x = 0$
$x = 0$, $x = 1$

singular points: $f'(x)$ exists everywhere.

Critical points: $x = -1/2, 0, 1, 2$

$f(x) = -2x^3 + 3x^2$

$f(-1/2) = -2 \cdot -\frac{1}{8} + 3 \cdot \frac{1}{4} = \underline{1}$

$f(0) = 0$

$f(1) = -2 + 3 = 1$

$f(2) = -2 \cdot 8 + 3 \cdot 4 = -16 + 12 = -4$

$\Rightarrow$ max value of $\underline{1}$ @ $x = -1/2$ and ~~or~~ $x = 1$

min value of -4 @ $x = 2$

Example (Example 3.1.4)

Find the maximum and minimum values of $f(x) = x^{2/3}$ on $[-1, 2]$. Also identify the x locations corresponding to these extreme values.
(Ans: Min value 0 at $x = 0$, max value $\sqrt[3]{4}$ at $x = 2$.)

f continuous, I closed.

endpoints: $x = -1, 2$

stationary points: $f'(x) = \frac{2}{3} x^{-1/3} = 0$ (no stationary points)

singular points: $f'(x) = \frac{2}{3} \frac{1}{x^{1/3}}$ DNE when $x = 0$

critical points: $x = -1, 0, 2$

$f(-1) = (-1)^{2/3} = 1$

$f(0) = 0$

$f(2) = 2^{2/3} > 1$

max value of $2^{2/3}$ @ $x = 2$

min valu of 0 @ $x = 0$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.