HW #7 due <u>Wedneseday</u>.

In-class review: Wednesday (Feb 26)

Lab: Thursday

Exam: Friday (No HW due this Friday)

Math 1210: Calculus I
Monotonicity and concavity

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.2

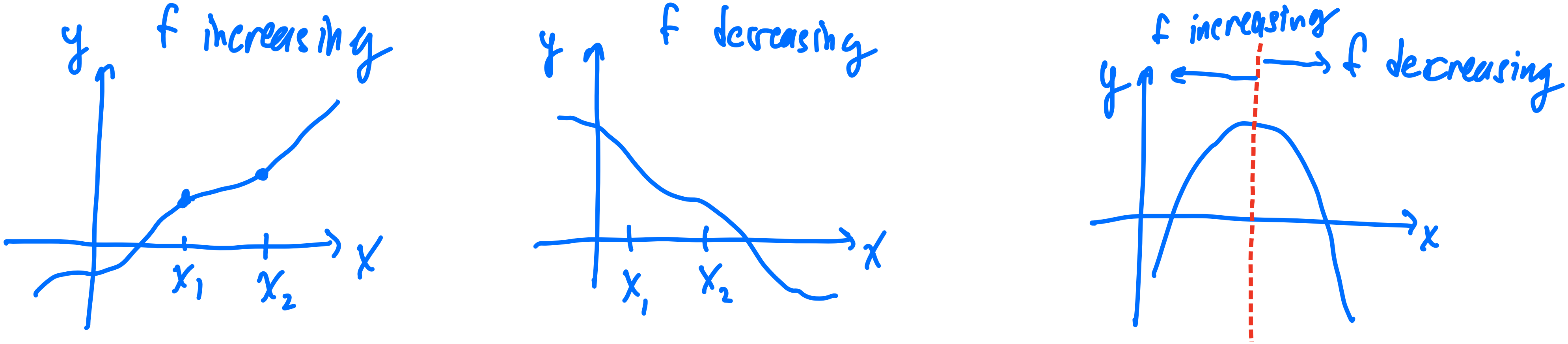
Monotonic functions

Derivatives can also be used to study *monotonicity* of functions.

Definition

Let f be a function defined on an interval I. Then:

- f is **increasing** on I if for every x_1, x_2 in I satisfying $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is **decreasing** on I if for every x_1, x_2 in I satisfying $x_1 < x_2$, then $f(x_1) > f(x_2)$.
- f is **strictly monotonic** on I if it's either increasing or decreasing on I.

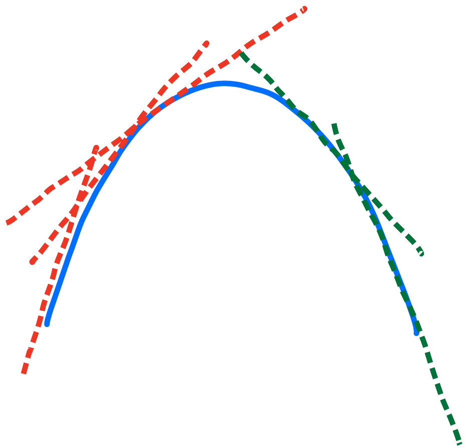


Monotonicity and derivatives

Since $f'(x)$ is the slope of the tangent line, and the tangent line approximates the graph of f around x:

If $f'(x)$ is positive, then $f(x)$ should be increasing.

If a tangent line looks like , then the function "should" be increasing around the tangent point.



Monotonicity and derivatives

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Theorem (Monotonicity theorem)

Assume f is continuous and differentiable on an interval I.

- *If $f'(x) > 0$ for all x in the interior of I, then f is increasing on I.*
- *If $f'(x) < 0$ for all x in the interior of I, then f is decreasing on I.*

"interior"? If $f'(x) > 0$ for x in $(0, 1)$, then f is increasing on $[0, 1]$.

If $f'(x) > 0$ for x in $(-3, -2)$, then f is increasing on $[-3, -2]$

Why? Suppose $f'(x) > 0$, i.e. $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c > 0$

i.e.: Suppose $h > 0$: $\frac{f(x+h) - f(x)}{h} \approx c > 0$

when h is close to 0

$\Longrightarrow f(x+h) - f(x) \approx ch > 0$

Example

If $f(x) = x^3 - 12x^2 + 45x + 3$, determine where f is increasing and where it's decreasing.
(Ans: Increasing on $(-\infty, 3]$ and $[5, \infty)$, and decreasing on $[3, 5]$.)

$$f'(x) = 3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x-3)(x-5)$$

Where is $f'(x) > 0$? Where is $f'(x) < 0$?

Note: $x > 3$: $(x-3) > 0$ $x > 5$: $(x-5) > 0$

$x < 3$: $(x-3) < 0$ $x < 5$: $(x-5) < 0$

I.e: stationary points for $f'(x)$ are where portions of $f'(x)$ change sign

$(x-3)$	$(-)$		$(+)$		$(+)$
$(x-5)$	$(-)$		$(-)$		$(+)$
$f'(x)$	$(+)$		$(-)$		$(+)$
x		3		5	

stationary points

$f'(x) < 0$ on $(3,5)$, so $f(x)$ is decreasing on $[3,5]$.

$f'(x) > 0$ on $(-\infty, 3)$, and $(5, \infty)$, so f is increasing on $(-\infty, 3]$ and $[5, \infty)$.

Monotonicity examples

D19-S04(b)

Example (Example 3.2.2)

Determine where $g(x) = \frac{x}{1+x^2}$ is increasing and where it is decreasing.
(Ans: Increasing on $[-1, 1\cancel{)}$], and decreasing on $(-\infty, -1]$ and $[1, \infty)$.)

$$g'(x) = \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}$$

Stationary points: $x = \pm 1$

Singular points: none

x	-2	-1	0	1	2
$f'(x)$	$(-)$		$(+)$		$(-)$

$$f'(-2) = \frac{1-(-2)^2}{(1+(-2)^2)^2} = \frac{-3}{25} < 0$$

$$f'(0) = \frac{1-0}{(1+0)^2} = 1 > 0$$

$$f'(2) = \frac{1-(2)^2}{(1+2^2)^2} = \frac{-3}{25} < 0$$

I.e.: f is increasing on $[-1, 1]$

f is decreasing on $(-\infty, -1]$ and $[1, \infty)$.

$(-\infty, -1] \cup [1, \infty)$

Concavity

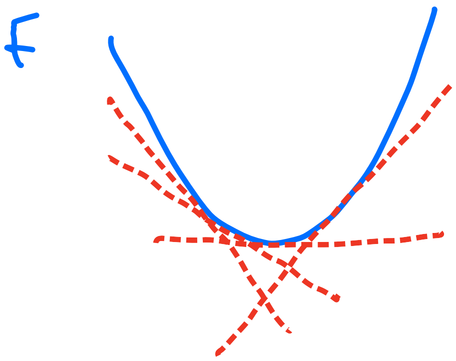
Understanding when $f(x)$ is increasing or decreasing is intuitive graphically.

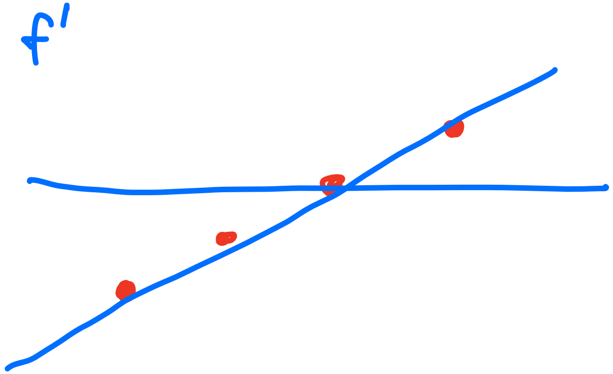
Less transparent, but still important, is understanding when $f'(x)$ is increasing or decreasing.

Definition

Suppose f is differentiable on an open interval I.

- The function f and its graph is **concave up** on I if $f'(x)$ is increasing on I.
- The function f and its graph is **concave down** on I if $f'(x)$ is decreasing on I.





Concavity

D19-S05(b)

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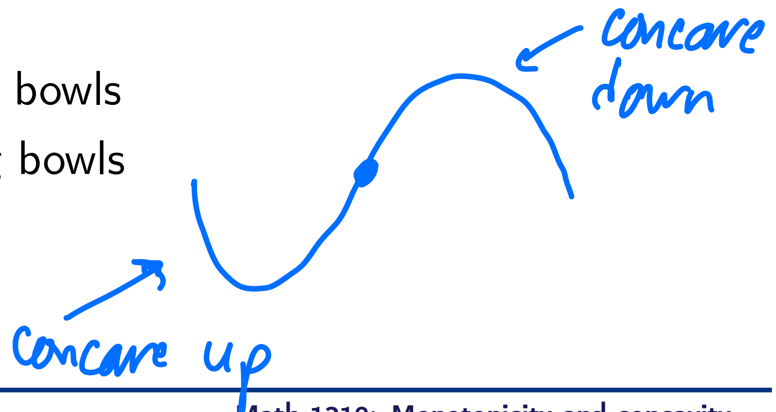
After some graphical experience, one gains some intuition about graphs of concave up or down functions:

- Functions that are concave up look like upward-facing bowls
- Functions that are down up look like downward-facing bowls



With monotonicity (say incrasing value of f), we determined that this could be determined through the sign of the first derivative.

Similarly, for concavity (say increasing value of f'), we can determine this through the sign of the second derivative.

Theorem

Suppose f is a function that has two derivatives ("twice differentiable") on an open interval I.

- *If $f''(x) > 0$ for all x in I, then f is concave up on I.*
- *If $f''(x) < 0$ for all x in I, then f is concave down on I.*

$f'(x) > 0 \rightarrow f$ increasing

$f''(x) > 0 \rightarrow f$ concave up

$f'(x) < 0 \rightarrow f$ decreasing

$f''(x) < 0 \rightarrow f$ concave down

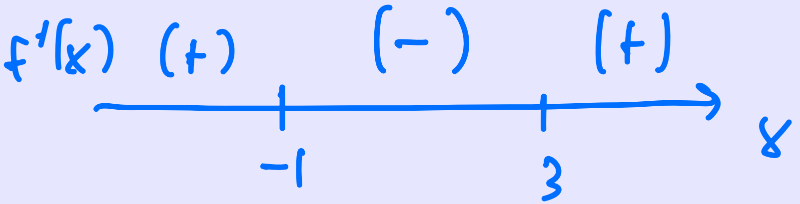
Example (Example 3.2.3)

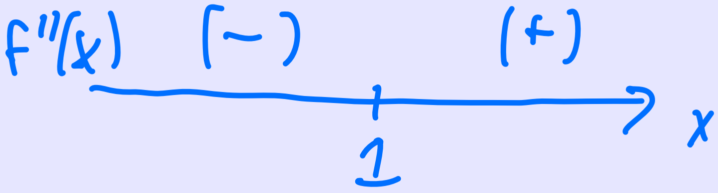
Let $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$. Determine where f is increasing, decreasing, concave up, and concave down.

(Ans: Increasing on $(-\infty, -1]$ and $[3, \infty)$, decreasing on $[-1, 3]$, concave up on $(1, \infty)$, concave down on $(-\infty, 1)$.)

$f'(x) = x^2 - 2x - 3 = (x-3)(x+1)$ ($x = -1, 3$ stationary pts)

$f''(x) = 2x - 2$ ($f''(x) = 0$ at $x = 1$)





$(x-3)$	$(-)$		$(-)$		$(+)$
$(x+1)$	$(-)$		$(+)$		$(+)$
$f'(x)$	$(+)$		$(-)$		$(+)$
x		-1		3	

f increasing on $(-\infty, -1]$ and $[3, \infty)$

f decreasing on $[-1, 3]$

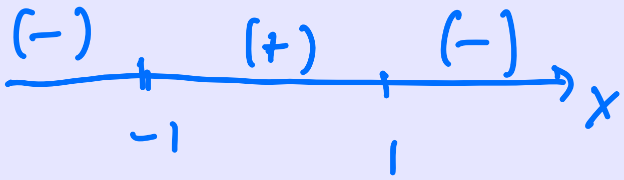
f concave up on $(1, \infty)$

f concave down $(-\infty, 1)$

Example (Example 3.2.3)

Let $g(x) = \frac{x}{1+x^2}$. Determine where g is concave up and and concave down. Use this information, and the information from a previous example, to sketch g.
(Ans: Concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$, concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.)

Recall: $g'(x) = \dfrac{1-x^2}{(1+x^2)^2}$



$$g''(x) = \frac{(-2x)(1+x^2)^{\cancel{2}\,1} - (1-x^2)\,2\,\cancel{(1+x^2)}\cdot 2x}{(1+x^2)^{\cancel{4}\,3}}$$

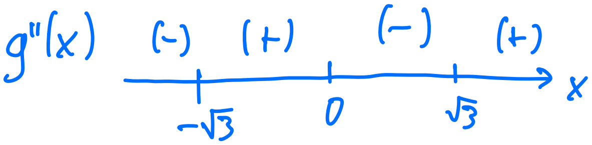
$$= \frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} = \frac{-2x - 2x^3 - 4x + 4x^3}{(1+x^2)^3}$$

$$= \frac{2x^3-6x}{(1+x^2)^3}$$

$g''(x)=0$ when $2x^3-6x=0$

$$2x(x^2-3)=0$$

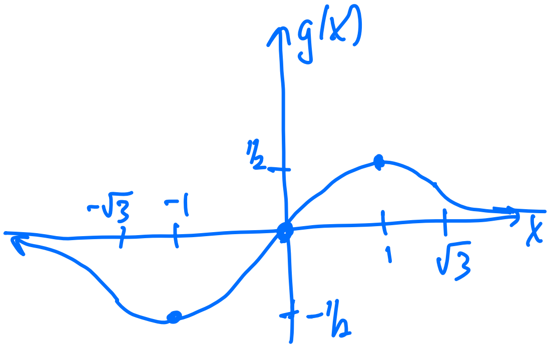
$$x=0,\ +\sqrt{3},\ -\sqrt{3}$$



g concave up on $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$

g concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$.

g increasing on $[-1, 1]$, decreasing elsewhere, $g(x)=\frac{x}{1+x^2}$

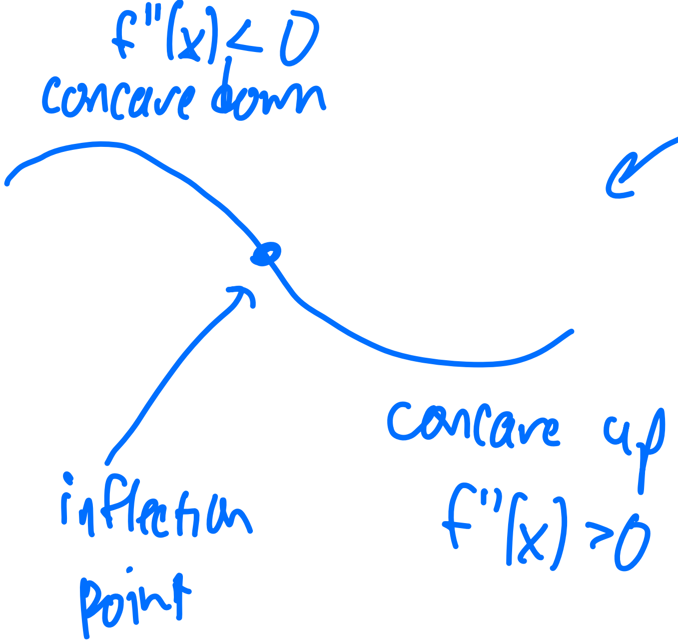


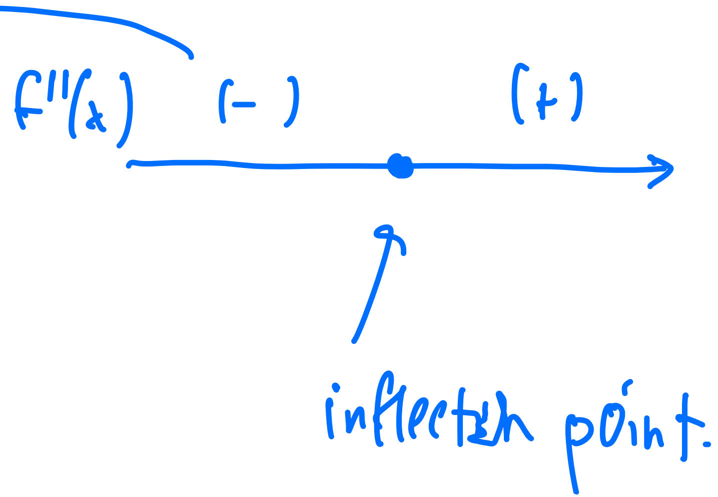
$$\lim_{x\to\pm\infty} g(x)=0$$

Inflection points

Points where concavity changes sign are graphically interesting: the graph changes from curving up/down to curving down/up.

Such points, say $x = c$, where $f''(x)$ changes sign are call **inflection points**.
(I.e., this means $f''(x)$ has one sign when $x < c$, and another sign when $x > c$.)





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Like minima and maxima, we need to identify candidates for inflection points before determining which points are inflections points.
Candidates for inflection points are:

- Points c where $f''(c) = 0$
- Points c where $f''(c)$ is not defined.

previous example: $x = 0, \pm\sqrt{3}$ are inflection points.

Example: inflection points

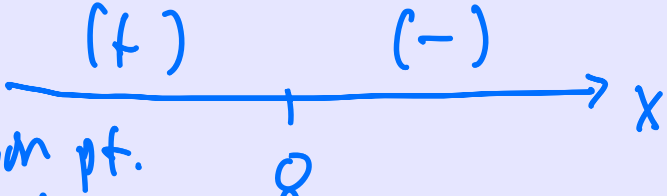
Example

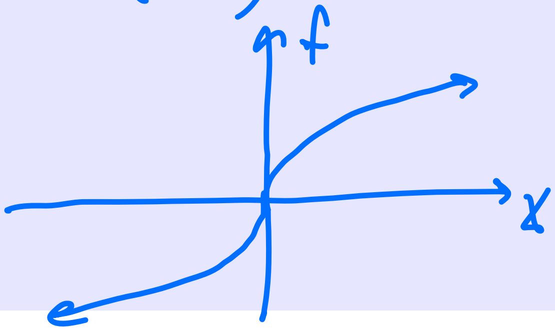
Determine all inflection points for $f(x) = x^{1/3}$.

$f'(x) = \frac{1}{3} x^{-2/3}$

$f''(x) = -\frac{2}{9} x^{-5/3} \longrightarrow$ candidate inflection point is $x=0$

(f'' undefined at $x=0$)





(0,0) is an inflection pt.

$x=0$ is an inflection point.

Midterm: I won't ask about inflection points.

Example: inflection points

Example

Suppose that $f(x)$ is a quadratic polynomial. Show that f has no inflection points.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.