Math 1210: Calculus I
Local extrema

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.3

Global and local extrema

Recall: a maximum value of f on some set S is the number $f(c)$ such that,

$$f(x) \leqslant f(c) \text{ for all } x \text{ in } S,$$

where the number c must also be in S.

Global and local extrema

Recall: a maximum value of f on some set S is the number $f(c)$ such that,

$$f(x) \leqslant f(c) \text{ for all } x \text{ in } S,$$

where the number c must also be in S.

Such a value might be called a **global maximum**, as it's the maximum value globally on S.

However, there are intuitively also locations that are **local maxima**.

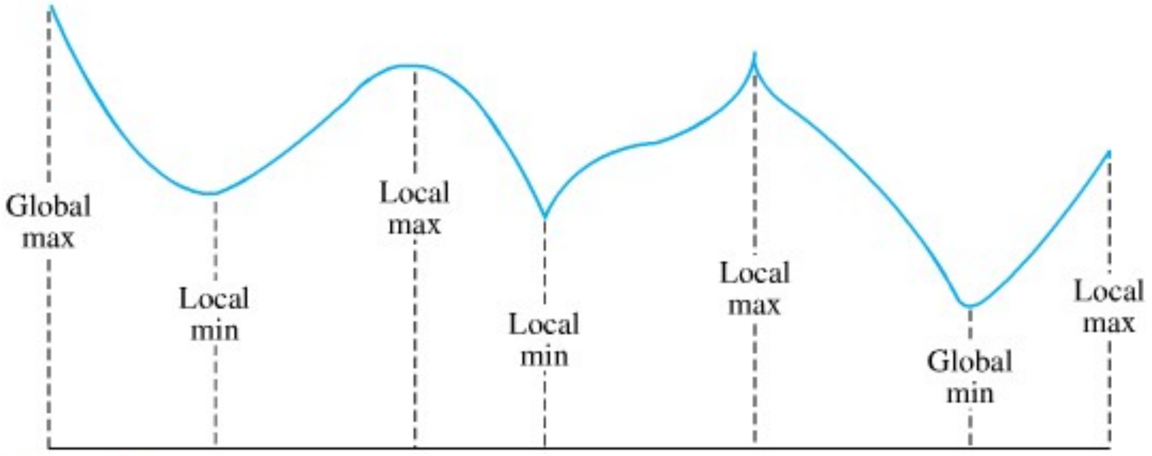


Figure 2

Local extrema

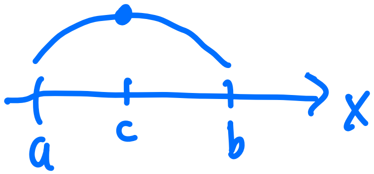
A local minimum or maximum can be defined as follows.

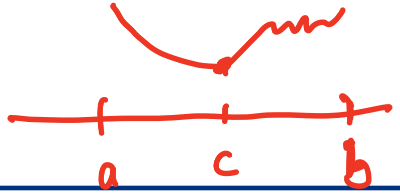
Definition

Suppose f is a function with domain S, and that c is some point in S.

- $f(c)$ is a **local maximum value** of f if there is some interval (a,b) containing c such that $f(x) \leqslant f(c)$ for every x in (a,b).
- $f(c)$ is a **local minimum value** of f if there is some interval (a,b) containing c such that $f(x) \geqslant f(c)$ for every x in (a,b).
- $f(c)$ is a **local extreme value** if it's either a local minimum or local maximum.

NB: global extrema are also local extrema.





Local extrema candidates

Like the global extremum case, candidates for x values where a local extremum may occur are largely determined by the derivative. The candidates are critical points:

- Points c corresponding to the endpoints of the domain
- Stationary points: points c such that $f'(c) = 0$.
- Singular points: points c such that $f'(c)$ is not defined.

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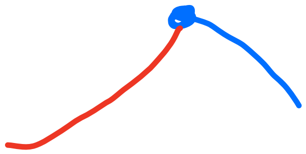
For the local extremum case, we shouldn't compute the maximum and minimum values of f over its critical points. (Because local extreme values need not be global extreme values.)

Instead, for a maximum we only need that f is decreasing to the right of $x = c$, and increasing to the left of $x = c$.



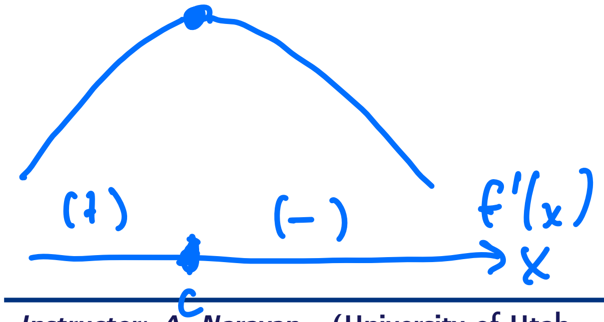
The first derivative test

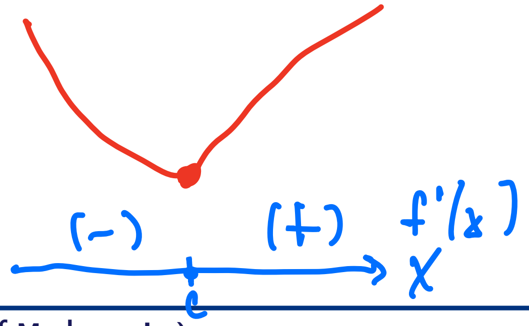
The test for local extremum can be determined by investigating the sign of the derivative around the critical point.

Theorem ("First derivative test")

Suppose f is a function on an interval (a, b) containing a critical point c.

- *If $f'(x) > 0$ for x in (a, c), and $f'(x) < 0$ for x in (c, b), then $f(c)$ is a local maximum for f.*
- *If $f'(x) < 0$ for x in (a, c), and $f'(x) > 0$ for x in (c, b), then $f(c)$ is a local minimum for f.*





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If $f'(x)$ has the same sign on either side of $x = c$, then $f(c)$ is not a local extremum.

Note that c may be a singular point ($f'(c)$ doesn't exist), but the above theorem is still true so long as the derivative is well-defined in the interval (a, b) without the point c.

First derivative test examples

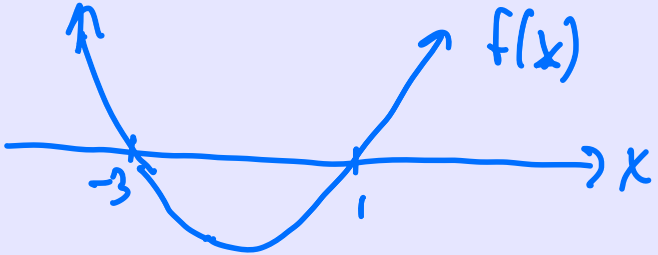
Example

Find the local extreme values of the function $f(x) = x^2 + 2x - 3$ on $(-\infty, \infty)$.
(Ans: $x = -1$ corresponds to a local minimum value of $f(-1) = -4$)

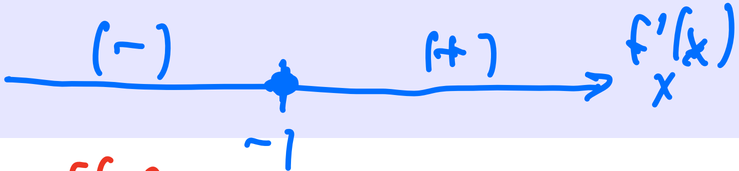
1.) No endpoints

2.) stationary points: $f'(x) = 2x+2 \rightarrow f'(x) = 0$ @ $x = -1$

3.) Singular points: none



$x=-1$ is the location of local min. $\rightarrow$ min value $f(-1) = 1-2-3 = -4$



First derivative test examples

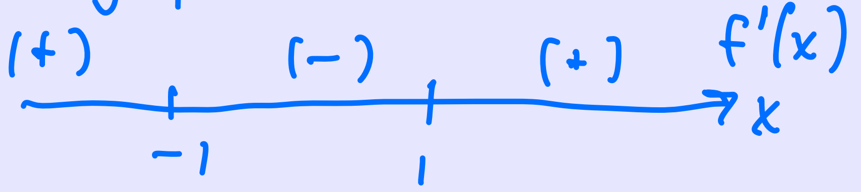
Example

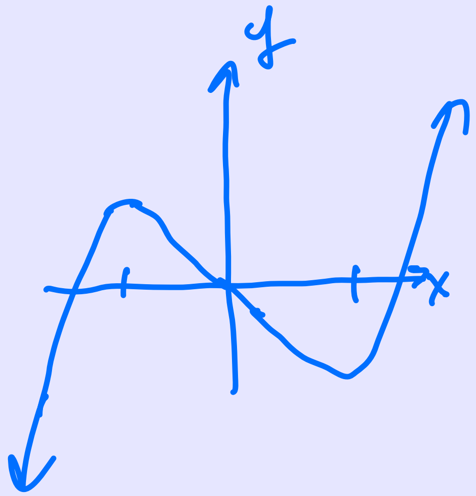
Find the local extreme values of the function $f(x) = x^3 - 3x$ on $(-\infty, \infty)$.
(Ans: $x = -1$ corresopnds to a local maximum value of $f(-1) = 2$, $x = 1$ is a local minimum value of $f(1) = -2$.)

Critical points: $f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$

No endpoints, no singular points

Stationary points: $f'(x) = 0$ @ $x = +1, -1$





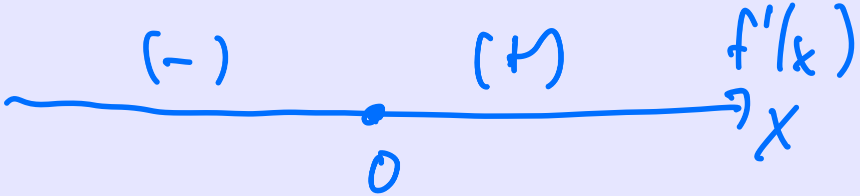
$\Rightarrow f(-1) = 2$ local max value, $f(+1) = -2$ local min value

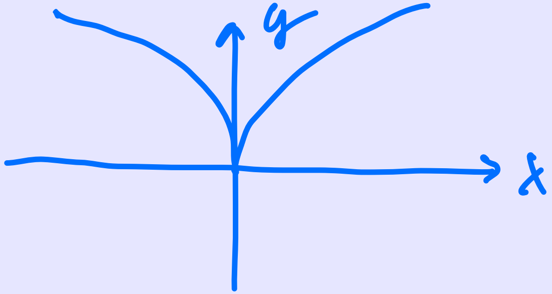
First derivative test examples

Example

Find the local extreme values of the function $f(x) = x^{2/3}$ on $(-\infty, \infty)$.
(Ans: $x = 0$ corresopnds to a local minimum value of $f(0) = 0$.)

$f'(x) = \frac{2}{3} x^{-1/3}$: one singular point @ $x = 0$





$f(0) = 0$ is a local minimum value

The second derivative test

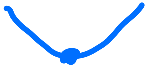
There is an alternative, sometimes easier, approach for identifying whether or not stationary points are local extrema.

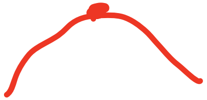
Theorem ("Second derivative test")

Suppose f is a twice differentiable function on (a, b), and that c is a stationary point for f on this interval.

$f'(c)=0$

- *If $f''(c) > 0$, then $f(c)$ is a local minimum value.*
- *If $f''(c) < 0$, then $f(c)$ is a local maximum value.*





$f(x)=x^3$

$f'(x)=3x^2 \qquad x=0$

$f''(x)=6x \qquad f''(0)=0$

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The above is only true for stationary points.
It is *not* apply for singular points (or endpoints of closed intervals).

First derivative test examples

Example

Find the local extreme values of the function $f(x) = x^2 + 2x - 3$ on $(-\infty, \infty)$ using the second derivative test.
(Ans: $x = -1$ corresponds to a local minimum value of $f(-1) = -4$)

$f'(x) = 2x + 2 \rightarrow x = -1$ is a stationary point

$f''(x) = 2 \rightarrow f''(-1) = 2 > 0 \Rightarrow x = -1$ is a local min
(by 2nd derivative test)

First derivative test examples

Example

Find the local extreme values of the function $f(x) = x^3 - 3x$ on $(-\infty, \infty)$ using the second derivative test.
(Ans: $x = -1$ corresopnds to a local maximum value of $f(-1) = 2$, $x = 1$ is a local minimum value of $f(1) = -2$.)

$f'(x) = 3x^2 - 3$ $x = \pm 1$ are stationary points.

$f''(x) = 6x \longrightarrow f''(+1) = 6 > 0 \Rightarrow x = +1$ is location of a local min

$f''(-1) = -6 < 0 \Rightarrow x = -1$ is location of a local max.

Global extrema on open intervals

In section 3.1, we discussed a procedure for computing global extrema on closed intervals. We have some tools to investigate this on half-/open intervals now.

Example (Example 3.3.6)

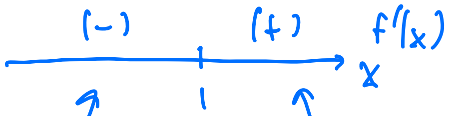
Find the maximum and minimum values (if they exist) of $f(x) = x^4 - 4x$ on $(-\infty, \infty)$.

Where are local minima/maxima?

Where is f increasing/decreasing?

$f'(x) = 4x^3 - 4 = 4(x-1)\underbrace{(x^2+x+1)}_{\text{positive everywhere}}$

$x=1$ is a stationary point



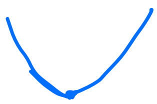
f decreasing

again, no max value for $x<1$

f increasing

for all $x>1$:
f can't have a global max.

first derivative test: $x=1$ is a local min.



for all $x>1$: f increasing
for all $x<1$: f decreasing

$\Big\}$ $x=1$ is location of a global min.

$\Rightarrow$ $f(1)$ is a global min value of f

$f(1) = 1-4 = -3$

(no global max)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.