Math 1210: Calculus I
Local extrema

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.3

Global and local extrema

Recall: a maximum value of f on some set S is the number $f(c)$ such that,

$$f(x) \leqslant f(c) \text{ for all } x \text{ in } S,$$

where the number c must also be in S.

Global and local extrema

Recall: a maximum value of f on some set S is the number $f(c)$ such that,

$$f(x) \leqslant f(c) \quad \text{for all } x \text{ in } S,$$

where the number c must also be in S.

Such a value might be called a **global maximum**, as it's the maximum value globally on S.

However, there are intuitively also locations that are **local maxima**.

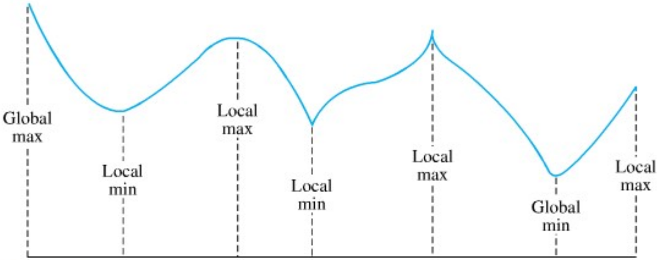

Figure 2

A local minimum or maximum can be defined as follows.

Definition

Suppose f is a function with domain S, and that c is some point in S.

- $f(c)$ is a **local maximum value** of f if there is some interval (a, b) containing c such that $f(x) \leqslant f(c)$ for every x in (a, b).
- $f(c)$ is a **local minimum value** of f if there is some interval (a, b) containing c such that $f(x) \geqslant f(c)$ for every x in (a, b).
- $f(c)$ is a **local extreme value** if it's either a local minimum or local maximum.

NB: global extrema are also local extrema.

Local extrema candidates

Like the global extremum case, candidates for x values where a local extremum may occur are largely determined by the derivative. The candidates are critical points:

- Points c corresponding to the endpoints of the domain
- Stationary points: points c such that $f'(c) = 0$.
- Singular points: points c such that $f'(c)$ is not defined.

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For the local extremum case, we shouldn't compute the maximum and minimum values of f over its critical points. (Because local extreme values need not be global extreme values.)

Instead, for a maximum we only need that f is decreasing to the right of $x = c$, and increasing to the left of $x = c$.

The first derivative test

The test for local extremum can be determined by investigating the sign of the derivative around the critical point.

Theorem ("First derivative test")

Suppose f is a function on an interval (a, b) containing a critical point c.

- *If $f'(x) > 0$ for x in (a, c), and $f'(x) < 0$ for x in (c, b), then $f(c)$ is a local maximum for f.*
- *If $f'(x) < 0$ for x in (a, c), and $f'(x) > 0$ for x in (c, b), then $f(c)$ is a local minimum for f.*

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If $f'(x)$ has the same sign on either side of $x = c$, then $f(c)$ is not a local extremum.

Note that c may be a singular point ($f'(c)$ doesn't exist), but the above theorem is still true so long as the derivative is well-defined in the interval (a, b) without the point c.

First derivative test examples

Example

Find the local extreme values of the function $f(x) = x^2 + 2x - 3$ on $(-\infty, \infty)$.
(Ans: $x = -1$ corresponds to a local minimum value of $f(-1) = -4$)

First derivative test examples

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Find the local extreme values of the function $f(x) = x^3 - 3x$ on $(-\infty, \infty)$.
(Ans: $x = -1$ corresopnds to a local maximum value of $f(-1) = 2$, $x = 1$ is a local minimum value of $f(1) = -2$.)

First derivative test examples

Example

Find the local extreme values of the function $f(x) = x^{2/3}$ on $(-\infty, \infty)$.
(Ans: $x = 0$ corresesopnds to a local minimum value of $f(0) = 0$.)

The second derivative test

There is an alternative, sometimes easier, approach for identifying whether or not stationary points are local extrema.

Theorem ("Second derivative test")

Suppose f is a twice differentiable function on (a, b), and that c is a stationary point for f on this interval.

- *If $f''(c) > 0$, then $f(c)$ is a local minimum value.*
- *If $f''(c) < 0$, then $f(c)$ is a local maximum value.*

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The above is only true for stationary points.
It is *not* apply for singular points (or endpoints of closed intervals).

Example

Find the local extreme values of the function $f(x) = x^2 + 2x - 3$ on $(-\infty, \infty)$ using the second derivative test.
(Ans: $x = -1$ corresponds to a local minimum value of $f(-1) = -4$)

Example

Find the local extreme values of the function $f(x) = x^3 - 3x$ on $(-\infty, \infty)$ using the second derivative test.
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Global extrema on open intervals

In section 3.1, we discussed a procedure for computing global extrema on closed intervals. We have some tools to investigate this on half-/open intervals now.

Example (Example 3.3.6)

Find the maximum and minimum values (if they exist) of $f(x) = x^4 - 4x$ on $(-\infty, \infty)$.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.