Math 1210: Calculus I
Graphing functions with calculus

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.5

Plotting functions

When plotting a function $y = f(x)$, there are some tools we had before considering calculus:

- Determine domain and range for f, e.g. $f(x) = \sqrt{x}$
- Investigate symmetry (odd/even functions), e.g. $f(x) = x^4 - x^2$
- Determine intercepts ($x=0$, $y=0$)
- Plot a few points
- (Maybe) Determine asymptotes

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With calculus, we have several more tools:

- Determine asymptotes
- Monotonicity: identify critical points, assess where f is increasing/decreasing
- Local maxima/minima: first/second derivative test analysis of critical points
- Concavity: analyze second derivative, find points of inflection

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With calculus, we have several more tools:

- Determine asymptotes (through limits)
- Monotonicity: identify critical points, assess where f is increasing/decreasing
- Local maxima/minima: first/second derivative test analysis of critical points
- Concavity: analyze second derivative, find points of inflection

The above is essentially a(n algorithmic) laundry list of things to consider when plotting a function.

Example (Example 3.5.1)

Sketch the graph of $f(x) = \frac{3x^5 - 20x^3}{32}$.

$$f(-x) = \frac{-(3x^5) + 20x^3}{32} = -f(x) \quad \text{(odd function)}$$

domain, range: $(-\infty, \infty)$

intercepts: $x = 0 \implies f(0) = 0$, $(0,0)$

$$y = 0 \implies 3x^5 - 20x^3 = 0$$

$$x^3[3x^2 - 20] = 0 \rightarrow x = 0, \pm\sqrt{\frac{20}{3}}$$

Calculus: $f'(x) = \frac{1}{32}\left[15x^4 - 60x^2\right]$

$$= \frac{15}{32}x^2\left[x^2 - 4\right] = \frac{15}{32}x^2(x-2)(x+2)$$

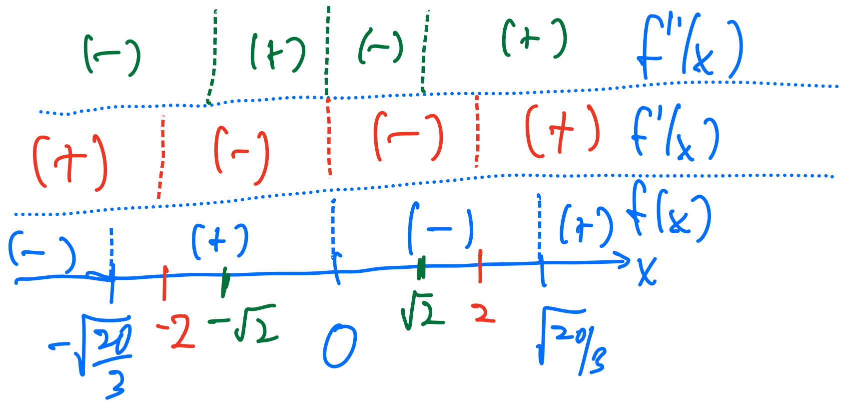
Critical pts: $x = 0, \pm 2$

$$f''(x) = \frac{d}{dx}\left[\frac{15}{32}\left(x^4 - 4x^2\right)\right] = \frac{15}{32}\left(4x^3 - 8x\right)$$

$$= \frac{15}{8}\left(x^3 - 2x\right)$$

$$= \frac{15}{8}x\left(x+\sqrt{2}\right)\left(x-\sqrt{2}\right)$$

$f''(x) = 0$ @ $x = 0, \pm\sqrt{2}$

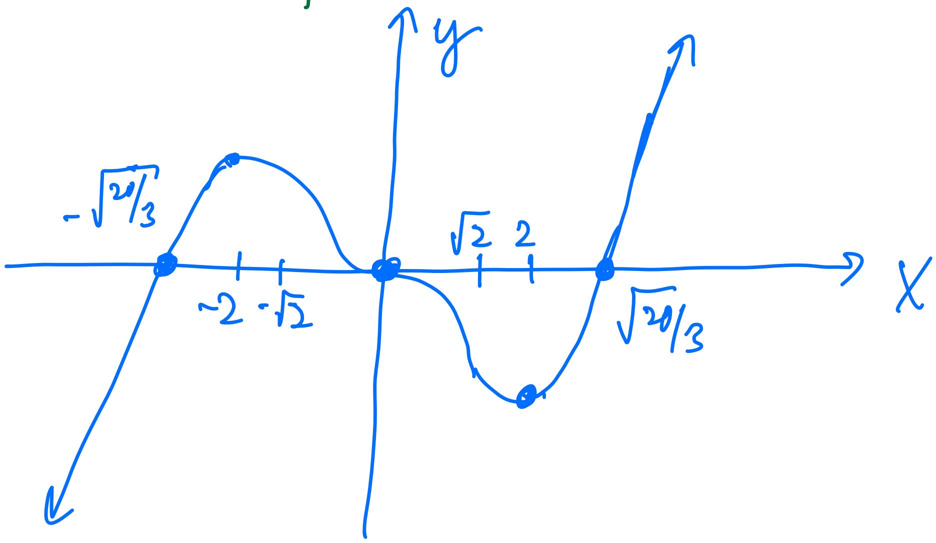


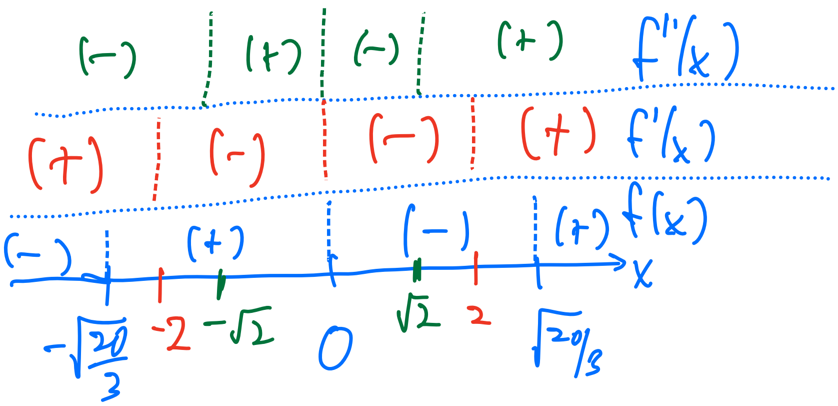
$$f(x) = \frac{1}{32}\left(3x^5 - 20x^3\right)$$

$x = -2$: location of local max

$x = +2$: location of local min

$x = 0, \pm\sqrt{2}$: points of inflection





Example (Problem 3.5.1)

Sketch the graph of $f(x) = x^3 - 3x + 5$.

Neither odd or even

$$\lim_{x \to \infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

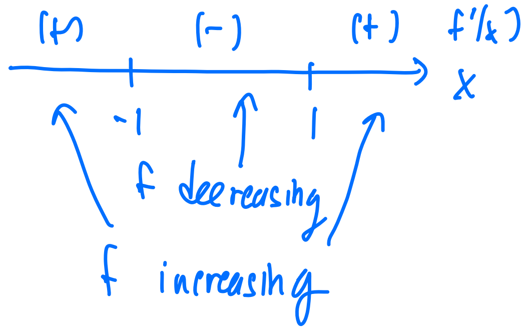
$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1)$$

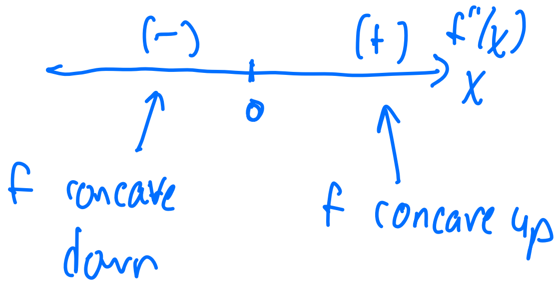
$$f''(x) = 6x$$

Critical points: $f'(x) = 0$, DNE

$x = \pm 1$

Candidate for inflection point: $x = 0$





first derivative test: local max @ $x = -1$

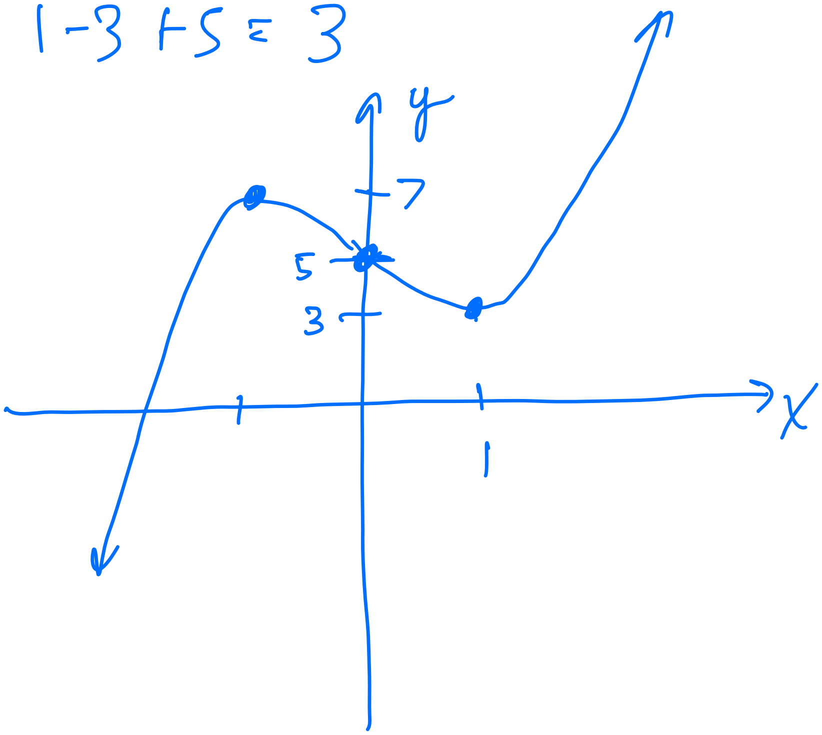
local min @ $x = +1$

$x = 0$: an inflection point (f'' changes sign)

$f(0) = 5$ $\quad$ ($f(x) = x^3 - 3x + 5$)

$f(-1) = -1 + 3 + 5 = 7$

$f(1) = 1 - 3 + 5 = 3$



Example (Problem 3.5.13)

Sketch the graph of $f(x) = \frac{x}{x-1}$.

Neither odd nor even

y-intercept: $f(0) = \frac{0}{0-1} = 0 \Rightarrow (0,0)$

x-intercept: $f(x) = 0 \Longrightarrow x = 0 \Rightarrow (0,0)$

$$\left.\begin{array}{l} \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \frac{1}{1-\frac{1}{x}} = 1 \\ \lim\limits_{x\to-\infty} f(x) = 1 \end{array}\right\} \Rightarrow y = 1 \text{ is a horizontal asymptote}$$

$f(x) = \frac{x}{x-1}$

$$\left.\begin{aligned} &\lim_{x\to 1^+} \frac{x}{x-1} = \lim_{x\to 1^+} \frac{1}{x-1} = +\infty \\ &\lim_{x\to 1^-} \frac{x}{x-1} = -\infty \end{aligned}\right\} \text{ } x=1 \text{ is a vertical asymptote}$$

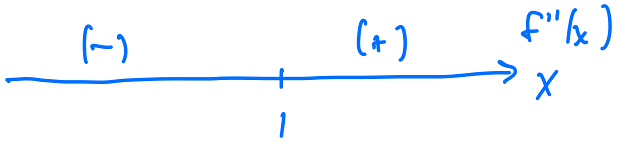
Calculus: $f'(x) = \frac{d}{dx}\left(\frac{x}{x-1}\right) = \frac{1(x-1) - x(1)}{(x-1)^2}$

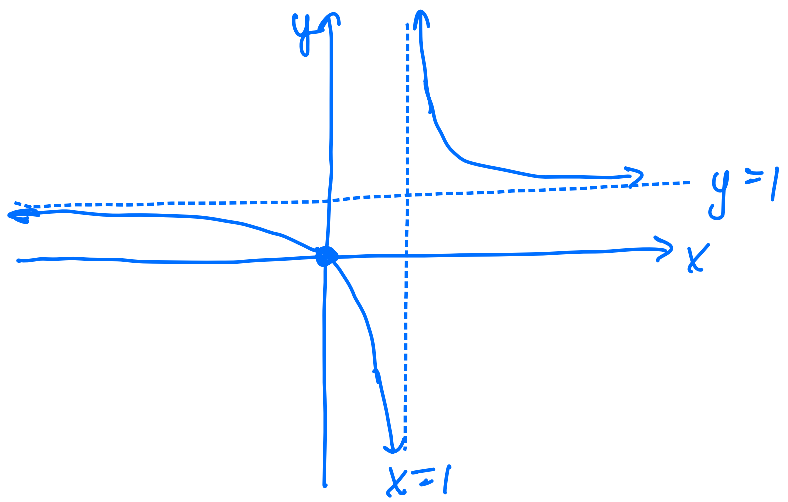
$= \frac{-1}{(x-1)^2} < 0 \quad (\text{if } x \neq 1)$

(i.e. f decreasing everywhere except at $x=1$)

$f''(x) = \frac{d}{dx}\left[-(x-1)^{-2}\right] = -(-2)(x-1)^{-3} \cdot \frac{d}{dx}(x-1)$

$= \frac{2}{(x-1)^3}$





Example (Example 3.5.4)

Sketch the graphs of $f(x) = x^{1/3}$ and $g(x) = x^{2/3}$.

We'll do $\underline{f(x)}$.

f is odd: $f(-x) = (-x)^{1/3} = -(x)^{1/3} = -f(x)$

$f(0) = 0$

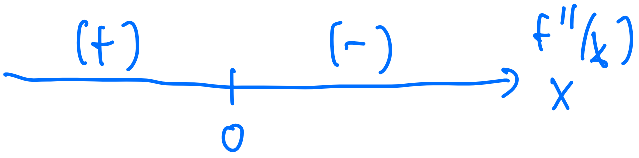
$f(1) = 1$

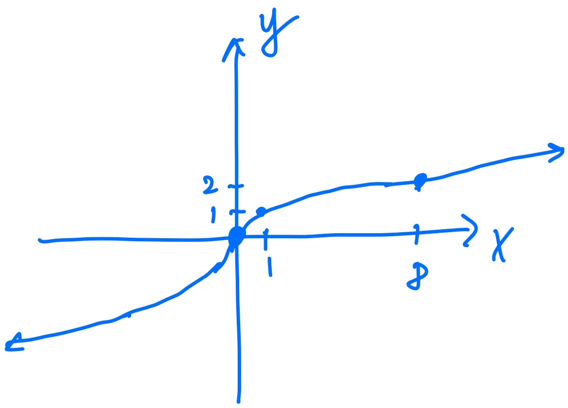
$f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} = \frac{1}{3(x^{1/3})^2} > 0$ (except at $x=0$)

f increasing everywhere (except at $x=0$)

$f''(x) = -\frac{2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}}$





References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.