Math 1210: Calculus I
Practical problems

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.4

Practical problems

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Step 1: Draw a picture for the problem and assign appropriate variables to the important quantities.

Step 2: Write a formula for the objective function Q to be maximized or minimized in terms of the variables from step 1.

Step 3: Use the conditions of the problem to eliminate all but one of these variables, and thereby express Q as a function of a single variable.

Step 4: Find the critical points (end points, stationary points, singular points).

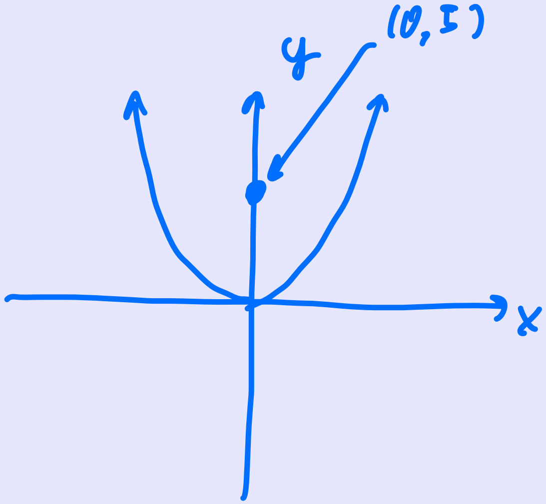
Step 5: Either substitute the critical values into the objective function or use the theory from the last section (i.e., the First and Second Derivative Tests) to determine the maximum or minimum.

Despite the steps above, there really isn't a recipe to solving these problems.

Examples

Example (Problem 3.4.5)

Find the points on the parabola $y = x^2$ that are cloest to the point $(0, 5)$. *Hint*: Minimize the square of the distance between (x, y) and $(0, 5)$.



distance between $(0,5)$ and a point (x,y):

$$d^2 = (x-0)^2 + (y-5)^2$$

x and y are ~~to~~ lie on parabola $\Rightarrow y = x^2$

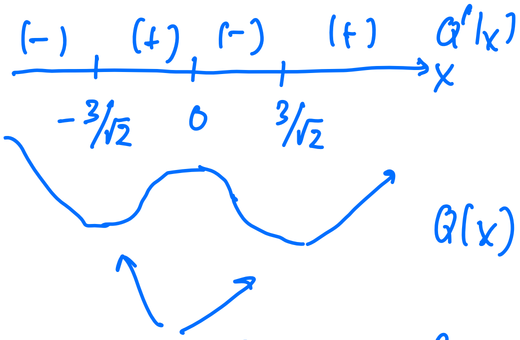
$$Q = d^2 = x^2 + (y-5)^2 = x^2 + (x^2-5)^2$$

minimize $Q(x) = x^2 + (x^2-5)^2$ for x in $(-\infty, \infty)$

$$Q'(x) = 2x + 2(x^2-5)\cdot 2x = 2x[1+2(x^2-5)]$$
$$= 2x[2x^2-9]$$

Stationary points: $x=0$, $2x^2-9=0$

$x = \pm 3/\sqrt{2}$



$x = \pm 3/\sqrt{2}$ are local minima

smallest value of Q between these must be a global minimum (because $Q(x) \to +\infty$ when $x \to \pm\infty$)

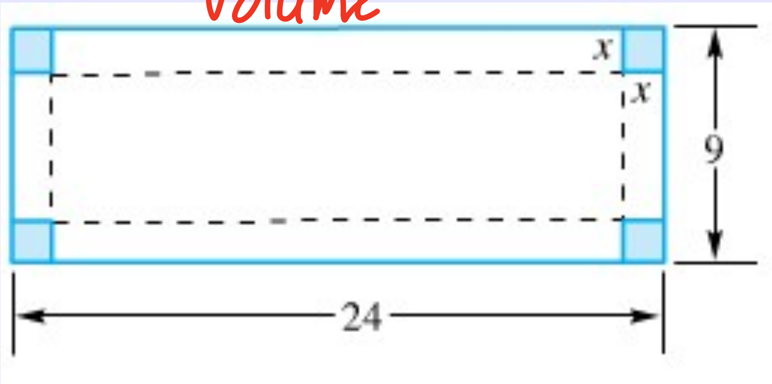
$$Q(x) = x^2 + (x^2-5)^2 \implies Q(+3/\sqrt{2}) = Q(-3/\sqrt{2})$$

$\implies x = \pm 3/\sqrt{2}$ are x-coordinates for closest points

Closest points are $(3/\sqrt{2}, 9/2)$ and $(-3/\sqrt{2}, 9/2)$

Example (Example 3.4.1)

A rectangular box is to be made from a piece of cardboard 24 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides, as in the figure. Find the dimensions of the box of maximum ~~value~~ volume. What is this volume?



Box bottom has length $24-2x$, width $9-2x$.

height of box is x.

Volume $Q(x) = x(9-2x)(24-2x)$
$= 2x(9-2x)(12-x)$
$= 2x(2x^2 - 33x + 108)$
$= 4x^3 - 66x^2 + 216x$

want to maximize $Q(x)$.

x can't be negative, and $x \leq 4.5$ (so that $9-2x \geq 0$)

Find ~~min~~ max value of $Q(x) = 4x^3 - 66x^2 + 216x$ for x inside $[0, 4.5]$.

Find critical points:

- endpoints: $x = 0,\ 4.5$
- singular points: $Q'(x) = 12x^2 - 132x + 216$ exists everywhere. (no singular points)
- stationary points: $Q'(x) = 0$

$$12x^2 - 132x + 216 = 0$$
$$12[x^2 - 11x + 18] = 0$$

$12(x-2)(x-9)=0$

$x=2, \cancel{9}$ 9 not in $[0, 4.5]$

Critical points: $x=0, 2, 4.5$.

$Q(x)=x(24-2x)(9-2x)$

$Q(0)=Q(4.5)=0$

$Q(2)=2(24-4)(9-4)$

$=2(20)(5)=200$ [in^3]

max value

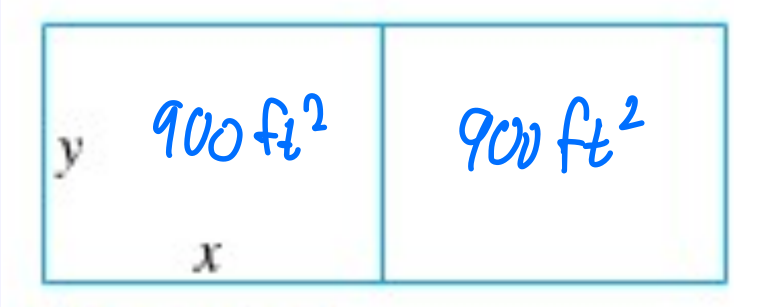
Box of dimensions $2 \times (9-4) \times (24-4)$,

i.e. $2 \times 5 \times 20$, is

box of max volume, 200 in^3.

Example (Problem 3.4.13)

A farmer wishes to fence off two identical adjoining rectangular pens, each with 900 square feet of area, as ~~sown~~ shown in the figure below. What are x and y so that the least amount of fence is required?



$xy = 900$

$\text{Perimeter} = 4x + 3y$

area constraint: $y = 900/x$

Perimeter is $4x + 3y = 4x + 3\left(900/x\right) = 4x + 2700/x$

$x < 0$ disallowed, any $x \geq 0$ allowed

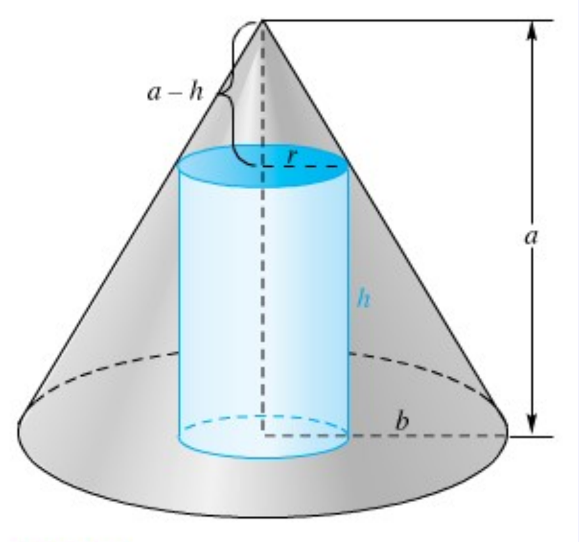
$x = 0$ disallowed ($xy \neq 900$)

minimize $Q(x) = 4x + 2700/x$ for x in $(0, \infty)$

Examples

Example (Example 3.4.3)

Find the dimensions of the right circular cylinder of greatest volume that can be inscribed in a given right circular cone.



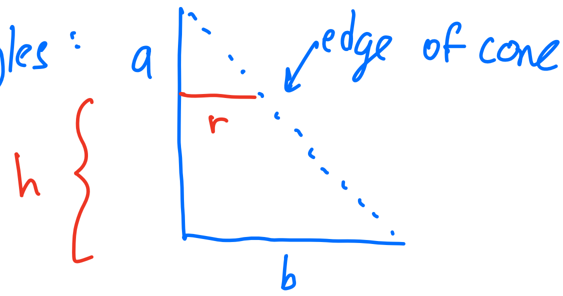
a, b: "given" constants

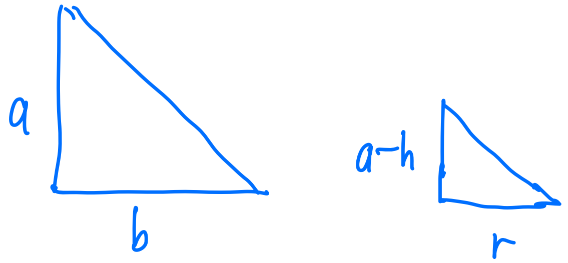
h, r: unknown

volume of cylinder:

$\pi r^2 \cdot h$

Can identify similar triangles:





these triangles have the same angles $\Rightarrow$ are similar

$$\frac{a-h}{r} = \frac{a}{b}$$ (constraint between r and h)

$$\Rightarrow r = \frac{b}{a}(a-h)$$

volume of cylinder is $\pi r^2 h = \pi\left(\frac{b}{a}(a-h)\right)^2 \cdot h$

constraints on h: $h \geq 0$, $h \leq a$

maximize $Q(h) = \pi\left(\frac{b}{a}\right)^2 (a-h)^2 \cdot h$ for h in $[0, a]$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.