Math 1210: Calculus I
MVT: Derivatives

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.6

The Mean Value Theorem, motivated

The geometric motivation for the statement of the Mean Value Theorem is pictured below.

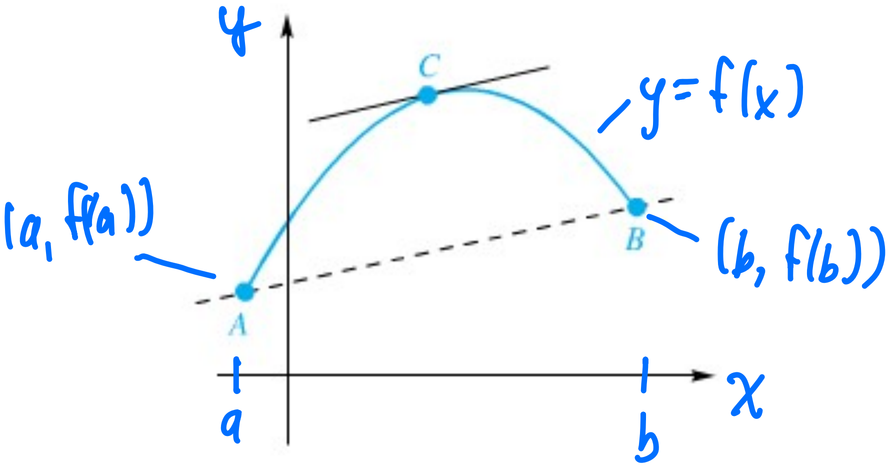

Figure 1

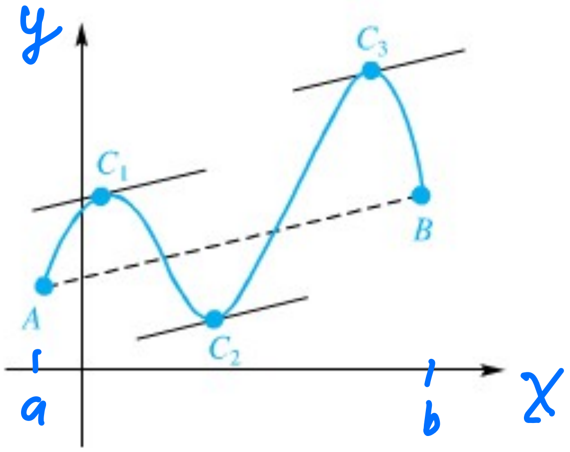

Figure 2

equation of dashed line: slope: $\dfrac{f(b)-f(a)}{b-a}$

point $(a, f(a))$

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

As drawn: Slope of tangent line at point C is same as $\dfrac{f(b)-f(a)}{b-a}$

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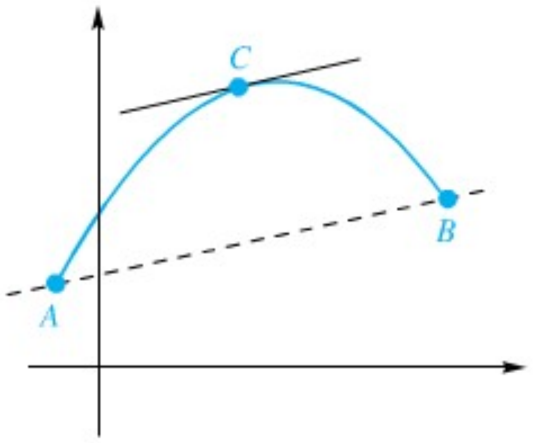


Figure 1

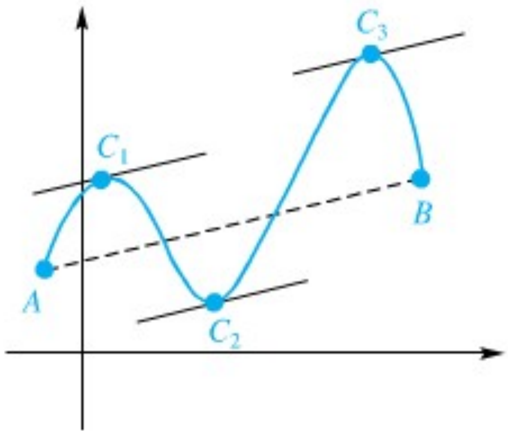


Figure 2

I.e.: If the graph of f is "nice" on an interval $[a, b]$, then:

The *slope* of the secant line connecting $(a, f(a))$ to $(b, f(b))$ must correspond to the slope of the tangent line to f at some point c in (a, b).

The Mean Value Theorem

We can make the statement precise, including describing the necessary assumption.

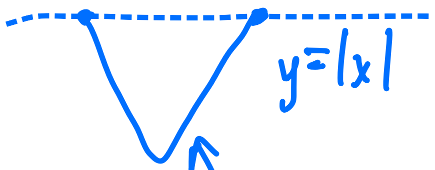
Theorem (Mean Value Theorem for Derivatives)

Assume f is a continuous function on the interval $[a, b]$, and is differentiable on (a, b). Then there exists at least one point c in (a, b) such that,

$$f'(c) = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{slope of secant line}} \quad \textit{or, equivalently} \quad f(b) - f(a) = f'(c)(b - a)$$

slope of tangent line

NB: It doesn't matter if $b < a$, the statement still holds for some c in (b, a).



|x| not differentiable at x=0, so MVT doesn't hold.

MVT proof

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for some } c \text{ in } (a, b)$$

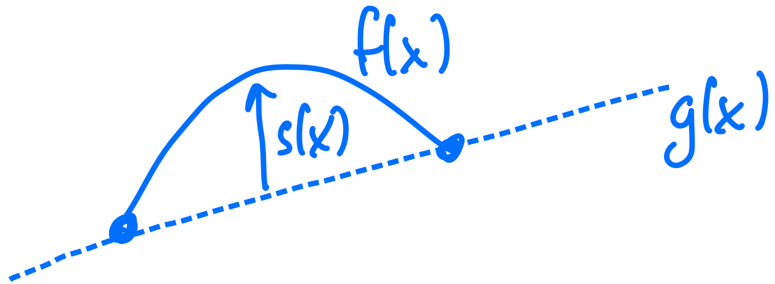
from first slide

The proof of the MVT is not too bad:

- The equation of the secant line is $g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x - a)$.
- The function $s(x) = f(x) - g(x)$ is a continuous function on $[a, b]$, and $s(a) = s(b) = 0$.
- s is a continuous function on $[a, b]$: it achieves its minimum and maximum
- We are looking for a location c in (a, b) where $s'(c) = 0$.

$$s'(c) = 0 \Rightarrow f'(c) = g'(c)$$

$$= \frac{f(b) - f(a)}{b-a}$$



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- If both extrema occur at endpoints, since $s(a) = s(b) = 0$, then $s(x) = 0$ for all x, so $s'(c) = 0$ everywhere in the interval.

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- s is a continuous function on $[a, b]$: it achieves its minimum and maximum
- We are looking for a location c in (a, b) where $s'(c) = 0$.
- If both extrema occur at endpoints, since $s(a) = s(b) = 0$, then $s(x) = 0$ for all x, so $s'(c) = 0$ everywhere in the interval.
- If not, then one extremum at some location $x = c$ occurs inside the interval, which must be a critical point. s is differentiable, so the only option is a stationary point: $s'(c) = 0$.

Example (Example 3.6.1)

Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt{x}$ on $[1, 4]$

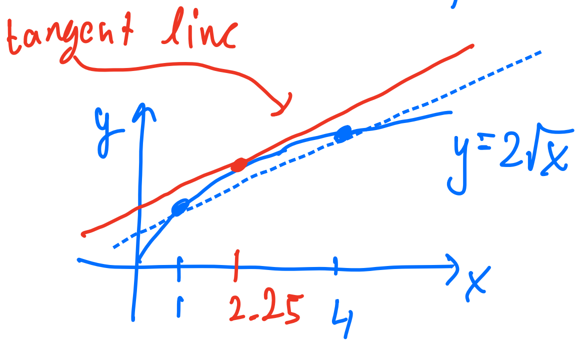
Note: f is continuous on $[1,4]$
f is differentiable on $(1,4)$

slope of secant line: $\frac{f(4)-f(1)}{4-1} = \frac{2\sqrt{4}-2\sqrt{1}}{4-1} = \frac{4-2}{3} = 2/3$

slope of tangent line: $f'(x) = \frac{d}{dx}[2\sqrt{x}] = \frac{d}{dx}[2x^{1/2}] = x^{-1/2}$
$= 1/\sqrt{x}$

MVT: $\frac{2}{3} = \frac{1}{\sqrt{x}}$ for some x in $(1,4)$

$x = 9/4 = 2.25$, inside $(1,4)$ ✓



Examples

Example (Example 3.6.2)

Let $f(x) = x^3 - x^2 - x + 1$ on $[-1, 2]$. Find all numbers c satisfying the conclusion ~~to~~ of the Mean Value Theorem.

f is continuous on $[-1,2]$

f is differentiable on $(-1,2)$

slope of secant line: $\frac{f(2)-f(-1)}{2--1} = \frac{(8-4-2+1)-(-1-1+1+1)}{3}$

$= \frac{3-0}{3} = 1$

slope of tangent line: $f'(x) = 3x^2 - 2x - 1$

MVT: $3x^2 - 2x - 1 = 1$ for some x in $(-1, 2)$

$3x^2 - 2x - 2 = 0$

$x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$ $\quad x = \frac{1}{6}\left[2 \pm \sqrt{4 + 24}\right]$

$= \frac{1}{6}\left(2 \pm 2\sqrt{7}\right) = \frac{1}{3}\left(1 \pm \sqrt{7}\right)$

$\frac{1+\sqrt{7}}{3} \sim \frac{3.\ldots}{3} \approx 1.\ldots$, inside $(-1, 2)$

$\frac{1-\sqrt{7}}{3} \sim \frac{-1.\ldots}{3}$, inside $(-1, 2)$

So $x = \frac{1}{3}\left(1 \pm \sqrt{7}\right)$ satisfy MVT.

Example (Example 3.6.3)

Let $f(x) = x^{2/3}$ on $[-8, 27]$. Show that the conclusion to the Mean Value Theorem fails and explain why.

Secant line slope: $\frac{f(27) - f(-8)}{27 - -8} = \frac{9-4}{35} = 1/7$

$f'(x) = 2/3\, x^{-1/3}$

when does $\frac{2}{3} x^{-1/3} = 1/7$?

$$x^{-1/3} = \frac{3}{14}$$

$$x^{1/3} = 14/3$$

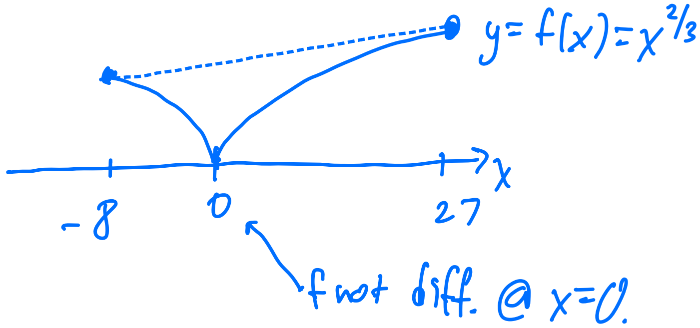
$$x = (14/3)^3$$

$$14/3 > 12/3 = 4$$

$\Rightarrow (14/3)^3 > 4^3 = 64$, not inside $(-8, 27)$

(MVT doesn't hold)

MVT doesn't hold because f is <u>not</u> differentiable inside $(-8, 27)$. $f'(x) = \frac{2}{3}x^{-1/3}$



More uses of the MVT: The Monotonicity Theorem

Recall: we know that if f is differentiable on an interval over which $f'(x) > 0$, then f is increasing on that interval.

The Mean Value Theorem explicitly shows us why:

- Let a, b with $a < b$ be any two points on the interval.
- By the MVT: $f(b) - f(a) = f'(c)\underbrace{(b-a)}_{b-a>0}$ for some c in (a, b).

$f'(c) > 0$

$\Rightarrow f(b) - f(a) = f'(c)(b-a) > 0 \Rightarrow f(b) > f(a)$

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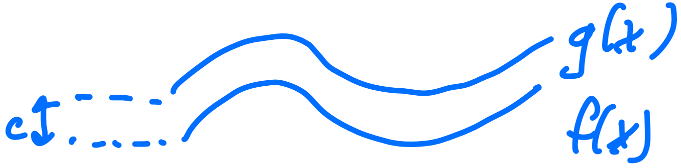
- Let a, b with $a < b$ be any two points on the interval.
- By the MVT: $f(b) - f(a) = f'(c)(b - a)$ for some c in (a, b).
- Since $b - a > 0$ and $f'(c) > 0$, then $f'(c)(b - a) > 0$, and hence $f(b) > f(a)$.
- I.e., for any a, b with $a < b$, then $f(a) < f(b)$, so f is increasing.

More uses of the MVT: Derivatives constrain functions, I

Note something straightforward: suppose f is a differentiable function, let c be a constant, and define

$$g(x) = f(x) + c.$$



A simple computation shows: $g'(x) = f'(x)$.

I.e., if functions differ simply by an additive constant, their derivatives are equal.

A harder question: If functions have equal derivative, do they differ simply by a constant?

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A harder question: If functions have equal derivative, do they differ simply by a constant?
The MVT furnishes a proof of this:

- Suppose $f'(x) = g'(x)$ on some interval I.
- Define $h(x) = f(x) - g(x)$, so that $h'(x) = f'(x) - g'(x) = 0$.
- Choose and fix any x_0 in I, and define $c = h(x_0)$.

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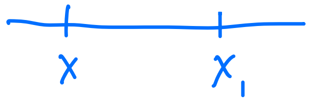
- Define $h(x) = f(x) - g(x)$, so that $h'(x) = f'(x) - g'(x) = 0$.
- Choose and fix any x_0 in I, and define $c = h(x_0)$.
- For arbitrary x in I, then the MVT states: $\cancel{H}(x) - \cancel{H}(x_1) = \cancel{H}'(c)(x - x_1)$ for some c inside the interval (x, x_1) or (x_1, x).
  $h'(c)$
  $h(x) - h(x_1) = h'(c)(x - x_1)$
- $h'(c) = 0$
  Since $\cancel{H}'(c) = 0$, this means $\cancel{H}(x) = \cancel{H}(x_1) = c$, and this is true for every x in the interval.
  h h
- I.e., $f(x) = g(x) + c$.

For arbitrary x in I, MVT: $h(x)-h(x_1)=h'(c)(x-x_1)$



But $h'(x)=0$ for all $x \Rightarrow h'(c)=0$

$\Rightarrow h(x)=h(x_1)$ (for every x)

$\Rightarrow f(x)-g(x)=\underbrace{h(x_1)}_{\text{"K"}}$ (for all x)

$\Rightarrow f(x)=g(x)+K.$

Theorem

Suppose $f'(x) = g'(x)$ for all x in (a, b). Then there is a constant c such that

$$f(x) = g(x) + c,$$

for all x in (a, b).

informally: given $f'(x)$, we know $f(x)$ up to an additive constant.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.