Math 1210: Calculus I
MVT: Derivatives

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.6

The Mean Value Theorem, motivated

The geometric motivation for the statement of the Mean Value Theorem is pictured below.

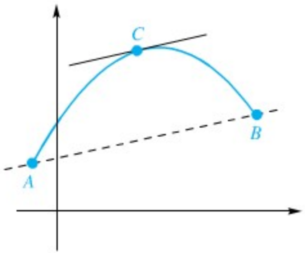

Figure 1

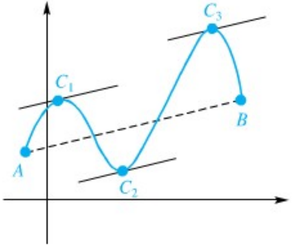

Figure 2

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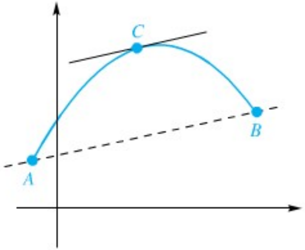

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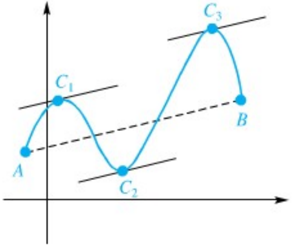

Figure 2

I.e.: If the graph of f is "nice" on an interval $[a, b]$, then:

The *slope* of the secant line connecting $(a, f(a))$ to $(b, f(b))$ must correspond to the slope of the tangent line to f at some point c in (a, b).

The Mean Value Theorem

We can make the statement precise, including describing the necessary assumption.

Theorem (Mean Value Theorem for Derivatives)

Assume f is a continuous function on the interval $[a, b]$, and is differentiable on (a, b). Then there exists at least one point c in (a, b) such that,

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \textit{or, equivalently} \quad f(b) - f(a) = f'(c)(b - a)$$

NB: It doesn't matter if $b < a$, the statement still holds for some c in (b, a).

MVT proof

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{for some } \ c \text{ in } (a, b)$$

The proof of the MVT is not too bad:

- The equation of the secant line is $g(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x - a)$.
- The function $s(x) = f(x) - g(x)$ is a continuous function on $[a, b]$, and $s(a) = s(b) = 0$.
- s is a continuous function on $[a, b]$: it achieves its minimum and maximum
- We are looking for a location c in (a, b) where $s'(c) = 0$.

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- If both extrema occur at endpoints, since $s(a) = s(b) = 0$, then $s(x) = 0$ for all x, so $s'(c) = 0$ everywhere in the interval.

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- We are looking for a location c in (a, b) where $s'(c) = 0$.
- If both extrema occur at endpoints, since $s(a) = s(b) = 0$, then $s(x) = 0$ for all x, so $s'(c) = 0$ everywhere in the interval.
- If not, then one extremum at some location $x = c$ occurs inside the interval, which must be a critical point. s is differentiable, so the only option is a stationary point: $s'(c) = 0$.

Example (Example 3.6.1)

Find the number c guaranteed by the Mean Value Theorem for $f(x) = 2\sqrt{x}$ on $[1, 4]$

Example (Example 3.6.2)

Let $f(x) = x^3 - x^2 - x + 1$ on $[-1, 2]$. Find all numbers c satisfying the conclusion to the Mean Value Theorem.

Example (Example 3.6.3)

Let $f(x) = x^{2/3}$ on $[-8, 27]$. Show that the conclusion to the Mean Value Theorem fails and explain why.

More uses of the MVT: The Monotonicity Theorem

Recall: we know that if f is differentiable on an interval over which $f'(x) > 0$, then f is increasing on that interval.

The Mean Value Theorem explicitly shows us why:

- Let a, b with $a < b$ be any two points on the interval.
- By the MVT: $f(b) - f(a) = f'(c)(b - a)$ for some c in (a, b).

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- Let a, b with $a < b$ be any two points on the interval.
- By the MVT: $f(b) - f(a) = f'(c)(b - a)$ for some c in (a, b).
- Since $b - a > 0$ and $f'(c) > 0$, then $f'(c)(b - a) > 0$, and hence $f(b) > f(a)$.
- I.e., for any a, b with $a < b$, then $f(a) < f(b)$, so f is increasing.

More uses of the MVT: Derivatives constrain functions, I

Note something straightforward: suppose f is a differentiable function, let c be a constant, and define

$$g(x) = f(x) + c.$$

A simple computation shows: $g'(x) = f'(x)$.

I.e., if functions differ simply by an additive constant, their derivatives are equal.

A harder question: If functions have equal derivative, do they differ simply by a constant?

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The MVT furnishes a proof of this:

- Suppose $f'(x) = g'(x)$ on some interval I.
- Define $h(x) = f(x) - g(x)$, so that $h'(x) = f'(x) - g'(x) = 0$.
- Choose and fix any x_0 in I, and define $c = h(x_0)$.

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- Define $h(x) = f(x) - g(x)$, so that $h'(x) = f'(x) - g'(x) = 0$.
- Choose and fix any x_0 in I, and define $c = h(x_0)$.
- For arbitrary x in I, then the MVT states: $H(x) - H(x_1) = H'(c)(x - x_1)$ for some c inside the interval (x, x_1) or (x_1, x).
- Since $H'(c) = 0$, this means $H(x) = H(x_1) = c$, and this is true for every x in the interval.
- I.e., $f(x) = g(x) + c$.

More uses of the MVT: Derivatives constrain functions, II

Theorem

Suppose $f'(x) = g'(x)$ *for all* x *in* (a, b)*. Then there is a constant* c *such that*

$$f(x) = g(x) + c,$$

for all x *in* (a, b)*.*

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.