Math 1210: Calculus I
Solving equations numerically

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.7

Solving equations

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- When f is a polynomial of degree n, there are at most n solutions. (There may be fewer.)
- For general, non-polynomial functions f, we generally don't know if a solution exists, or how many solutions exist.

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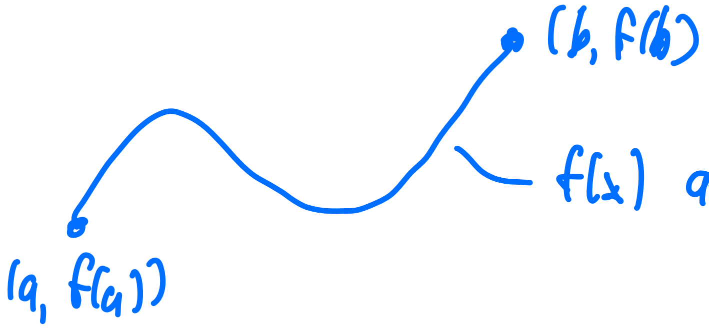
The main goal of this section: concepts we've learned (in particular, calculus) can be used in computational algorithms to determine solutions.

Method 1: Bisection

The method of **bisection** is an application of the *Intermediate Value Theorem*.
(Recall: if there is a continuous $f(x)$ on an interval $[a, b]$, then f attains every value between $f(a)$ and $f(b)$ somewhere inside the open interval (a, b).)



f(x) attains all values between f(a) and f(b).

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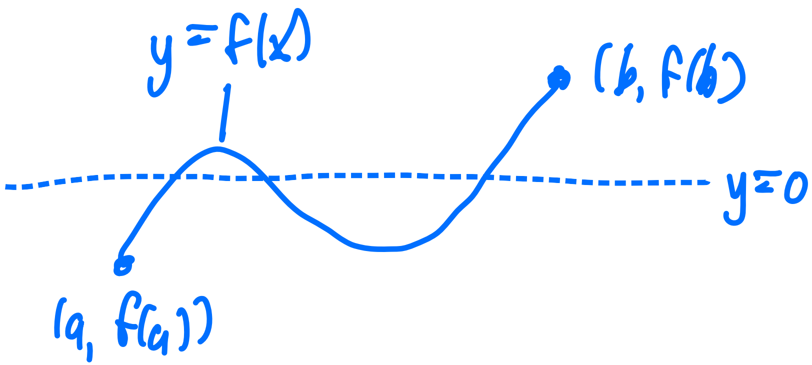
As before we wish to find a value of x such that $f(x) = 0$.
Suppose we know two values a, b with $a < b$ such that

$$f(a) < 0 < f(b) \quad \text{or} \quad f(b) < 0 < f(a).$$



intermediate value theorem:
$f(x) = 0$ for some x in (a, b).

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$$f(a) < 0 < f(b) \quad \text{or} \quad f(b) < 0 < f(a).$$

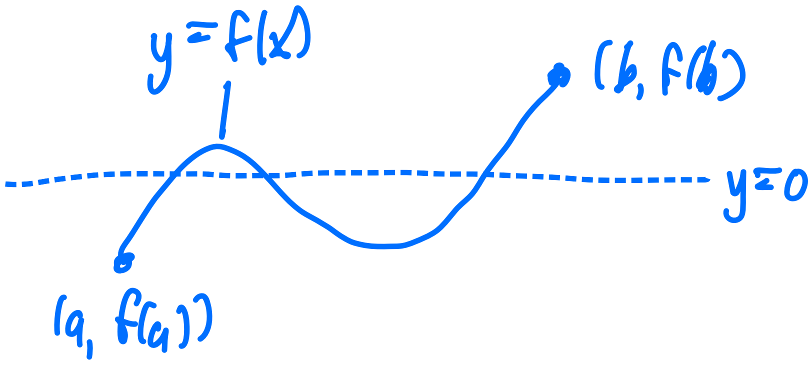
Then by the intermediate value theorem, there is at least one value of x in (a, b) such that $f(x) = 0$.

In exactly this scenario, the method of bisection helps us computationally identify this value.

Bisection explained

The above intuition leads to a fairly straightforward procedure:
Suppose we are given a function f, and an interval $[a, b]$ such that $f(a) < 0$ and $f(b) > 0$.
(If the roles of a and b are reversed, the procedure is largely the same.)

here: $f(a) < 0 < f(b)$



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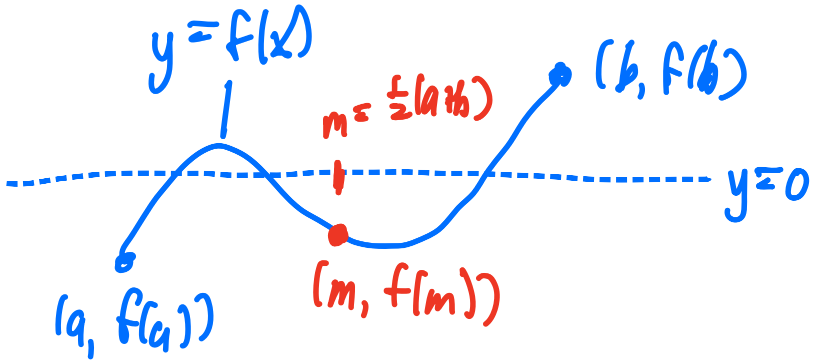
1. Define $m = \frac{1}{2}(a + b)$, the midpoint between a and b.
2. Evaluate $f(m)$



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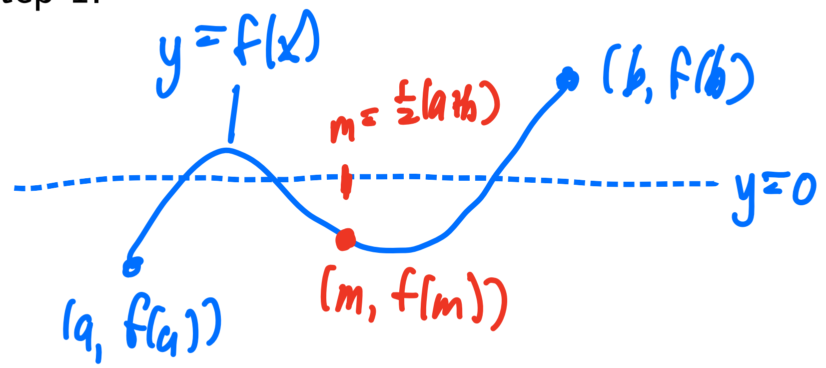
2. Evaluate $f(m)$
   - If $f(m) = 0$, we are done, return m.
   - If $f(m) < 0$, then $[m, b]$ is a new interval where f takes different signs at the endpoints. Set $a \leftarrow m$, $b \leftarrow b$, go back to step 1.
   - If $f(m) > 0$, then $[a, m]$ is a new interval where f takes different signs at the endpoints. Set $a \leftarrow a$, $b \leftarrow m$, go back to step 1.



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The above procedure typically doesn't terminate ($f(m) = 0$ exactly almost never happens).
One often *terminates* the procedure when the length of the interval $b - a$ is "sufficiently small", say is some value E.
The "proper" choice of E is an art.

Example (Example 3.7.1)

Determine the real root of $f(x) = x^3 - 3x - 5 = 0$ that lies inside the interval $[2, 3]$.

$a = 2$ $\quad f(a) = 8 - 6 - 5 = -3$ $\quad f(a) < 0 < f(b)$

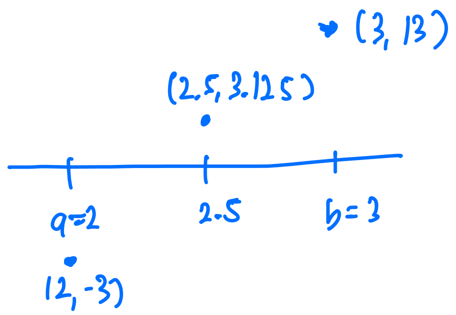
$b = 3$ $\quad f(b) = 27 - 9 - 5 = 13$

can use bisection ✓

Iteration 1: $m = \frac{1}{2}(a+b) = 2.5$

$f(m) = (2.5)^3 - 3 \cdot 2.5 - 5 = (2.5)^3 - 12.5$

$= 15.625 - 12.5 = 3.125$

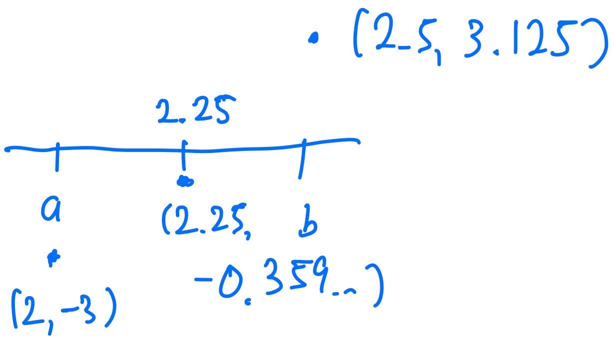


Since f(a) and f(m) have different signs, choose new interval as $[a, m] = [2, 2.5]$

Assign $[a, b] = [2, 2.5]$

Iteration 2: $m = \frac{1}{2}(a+b) = 2.25$

$$f(m) = (2.25)^3 - 3 \cdot 2.25 - 5 = (2.25)^3 - 11.75$$
$$= 11.390625 - 11.75$$
$$= -0.359\ldots$$



Since f(m) and f(b) have different signs, choose $[m, b]$ as next interval

Assign $[a, b] = [2.25, 2.5]$

Bisection example

Example (Example 3.7.1)

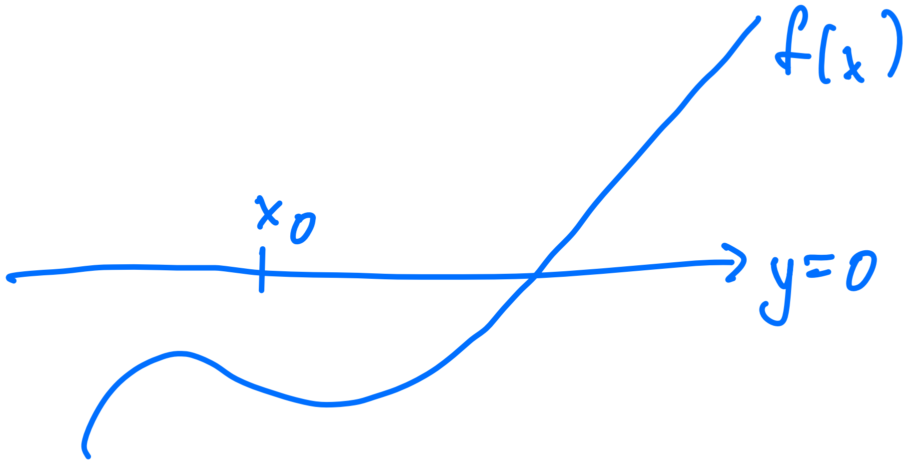
Determine the real root of $f(x) = x^3 - 3x - 5 = 0$ that lies inside the interval $[2, 3]$.

n	h_n	m_n	$f(m_n)$
1	0.5	2.5	3.125
2	0.25	2.25	−0.359
3	0.125	2.375	1.271
4	0.0625	2.3125	0.429
5	0.03125	2.28125	0.02811
6	0.015625	2.265625	−0.16729
7	0.0078125	2.2734375	−0.07001
8	0.0039063	2.2773438	−0.02106
9	0.0019531	2.2792969	0.00350
10	0.0009766	2.2783203	−0.00878
11	0.0004883	2.2788086	−0.00264
12	0.0002441	2.2790528	0.00043
13	0.0001221	2.2789307	−0.00111
14	0.0000610	2.2789918	−0.00034
15	0.0000305	2.2790224	0.00005
16	0.0000153	2.2790071	−0.00015
17	0.0000076	2.2790148	−0.00005
18	0.0000038	2.2790187	−0.000001
19	0.0000019	2.2790207	0.000024
20	0.0000010	2.2790197	0.000011
21	0.0000005	2.2790192	0.000005
22	0.0000002	2.2790189	0.0000014
23	0.0000001	2.2790187	−0.0000011
24	0.0000001	2.2790188	0.0000001

Method 2: Newton's Method

The previous method, bisection, did not exercise calculus.
This second approach explicitly uses the lesson's we've learned in this class:

Again, we wish to find a value of x such that $f(x) = 0$.
Assume we have an *initial guess* x_0.



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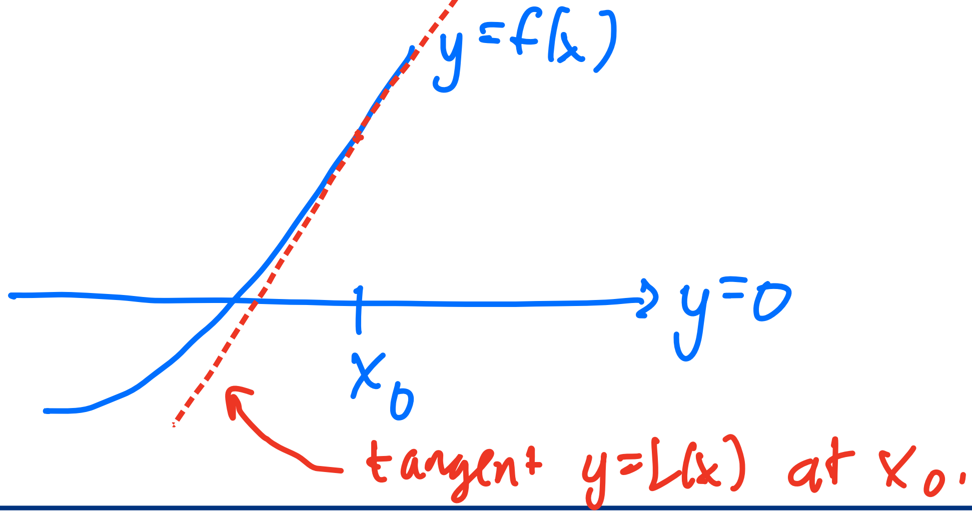
Again, we wish to find a value of x such that $f(x) = 0$.
Assume we have an *initial guess* x_0.

Recall that we can approximate graphs of curves with tangent lines.
This is useful because solving $f(x) = 0$ can be hard, but if $L(x)$ is a linear approximation, i.e., $L(x) = ax + b$, then solving $L(x) = 0$ is very easy.

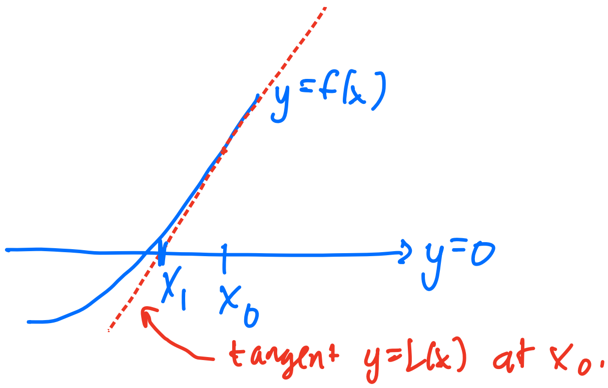


Since $f(x) \approx L(x)$ (near x_0), Let's solve $L(x)=0$ instead of $f(x)=0$.

Equation of tangent line at x_0: $y - f(x_0) = f'(x_0)(x - x_0)$

Solve for $y=0$: $0 - f(x_0) = f'(x_0)(x - x_0)$

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$



Let's call this approximate solution x_1:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Hope: x_1 is a good approximation to root of $f(x)$ (better than x_0)

Next guess: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

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Let's construct the linear approximation at our initial guess x_0.

The slope of the tangent line is $f'(x_0)$, and it passes through the point $(x_0, f(x_0))$.
Therefore, the equation of the line is:

$$y - f(x_0) = f'(x_0)(x - x_0) \quad \Longrightarrow \quad y = L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Newton's method

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Instead of solving $f(x) = 0$, we'll solve $L(x) = 0$ as an approximation. This is not too hard:

$$L(x) = 0 \quad \Longrightarrow \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

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Again, this value of x is a *guess*. It's not exact because f is (almost) never a linear function.

However, we expect it to be a better guess than x_0. So let's call this new guess x_1:

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As one expects, we can repeat this process:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots$$

This iterative algorithm is **Newton's method**.

Newton's method example

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2	2.30
3	2.2793
4	2.2790188
5	2.2790188

Geomtrically: Newton's method works

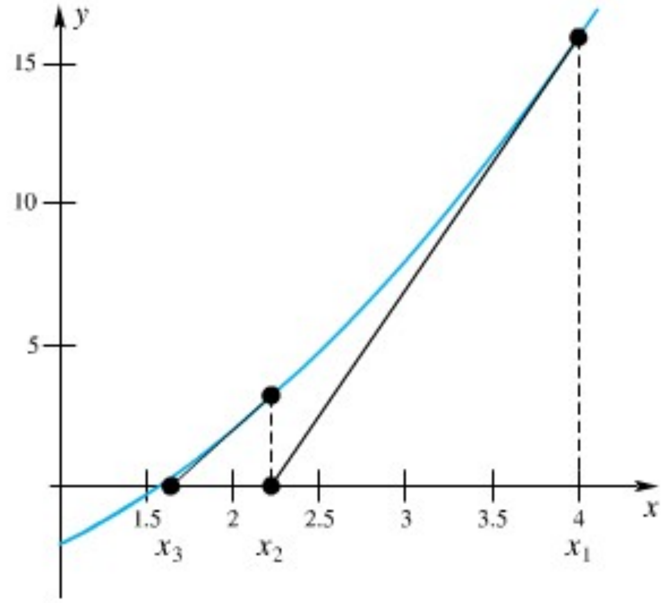

Figure 4

Geomtrically: Newton's method can fail, badly

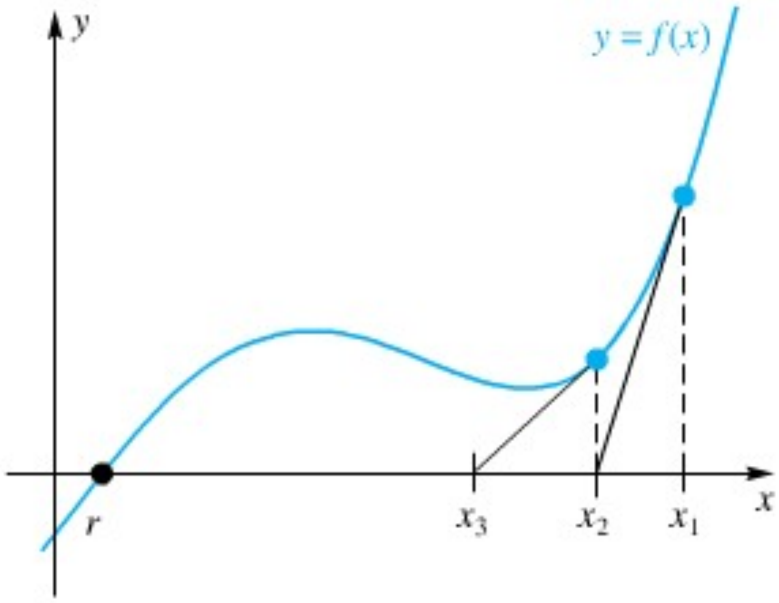

Figure 6

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.