Math 1210: Calculus I
Antiderivatives

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.8

"Undoing" differentiation

We've been discussing the task of differentiating or taking derivatives:

$$f(x) \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x},$$

and we've collected lots of rules and procedures for computing derivatives.

"Undoing" differentiation

We've been discussing the task of differentiating or taking derivatives:

$$f(x) \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x},$$

and we've collected lots of rules and procedures for computing derivatives.

Our next task will be to *invert* the differentiation process. (Just as subtraction inverts addition, division inverts multiplication, etc.)

I.e., we will investigate the task of performing the operation:

$$f'(x) \xrightarrow{\text{"invert" } \frac{\mathrm{d}}{\mathrm{d}x}} f(x)$$

"Undoing" differentiation

We've been discussing the task of differentiating or taking derivatives:

$$f(x) \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x},$$

and we've collected lots of rules and procedures for computing derivatives.

Our next task will be to *invert* the differentiation process. (Just as subtraction inverts addition, division inverts multiplication, etc.)

I.e., we will investigate the task of performing the operation:

$$f'(x) \xrightarrow{\text{"invert"} \frac{\mathrm{d}}{\mathrm{d}x}} f(x)$$

The task of inverting differentiation is called **antidifferentiation** or **integration**.

The function f is the **antiderivative** of f'.

A first observation

Recall: If two functions f and g are differentiable, then

$$f'(x) = g'(x) \quad \text{if and only if} \quad f(x) = g(x) + c,$$

for some constant c.

An immediate corollary: If $f(x)$ is an(y) antiderivative of f, then $f(x) + c$ is an(other) antiderivative of f.

A first observation

Recall: If two functions f and g are differentiable, then

$$f'(x) = g'(x) \quad \text{if and only if} \quad f(x) = g(x) + c,$$

for some constant c.

An immediate corollary: If $f(x)$ is an(y) antiderivative of f, then $f(x) + c$ is an(other) antiderivative of f.

Example (Example 3.8.1)

Find an antiderivative of $f(x) = 4x^3$ on $(-\infty, \infty)$.
(Ans: $F(x) = x^4$, or $F(x) = x^4 + 1$, or ...)

$F(x) = x^4 \implies dF/dx = 4x^3 = f(x)$, or $F(x) = x^4 + \pi$ ✓

or $F(x) = x^4 + c$ for any constant c.

"General" antiderivatives

If f is an antiderivative of f', then so is $f(x) + c$ for any constant c.
In particular, f is <u>an</u> antiderivative since it's not unique.

$\Longrightarrow$ $f(x)+c$ is the "family" of all possible antiderivatives.

Differentiation: there is a single derivative for a given function.

Integration: there are infinitely many antiderivatives

“General” antiderivatives

If f is an antiderivative of f', then so is $f(x) + c$ for any constant c.
In particular, f is an antiderivative since it's not unique.

In addition, if f is an antiderivative of f', then the only possible antiderivatives of f' are of the form $f(x) + c$ for some constant c.

"General" antiderivatives

If f is an antiderivative of f', then so is $f(x) + c$ for any constant c.
In particular, f is an antiderivative since it's not unique.

In addition, if f is an antiderivative of f', then the only possible antiderivatives of f' are of the form $f(x) + c$ for some constant c.

For this reason, if $f(x)$ is a(ny) antiderivative of $f'(x)$, then we call $f(x) + c$ for arbitrary constant c the **general antiderivative** of f'.

"General" is often omitted, and we simply say *the* antiderivative when referring to the general antiderivative.

Antiderivative notation

The notation we use for the derivative of f is f', or $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

The notation we use for the antiderivative of f is,

$$\int f(x) \mathrm{d}x$$

$\int f(x) dx$ → the "general antiderivative" of $f(x)$.

NB: The "$\mathrm{d}x$" is <u>not</u> optional, just as the "$\mathrm{d}x$" in $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not optional!

~~$\int f(x)$~~ $\int f(x) dx$

~~df~~ $\frac{df}{dx}$

Antiderivative notation

The notation we use for the derivative of f is f', or $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

The notation we use for the antiderivative of f is,

$$\int f(x) \mathrm{d}x$$

NB: The "$\mathrm{d}x$" is <u>not</u> optional, just as the "$\mathrm{d}x$" in $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not optional!

The symbol "∫" is an elongated/stylized "s". It stands for "sum", but we won't show why until later.

Antiderivative notation

The notation we use for the derivative of f is f', or $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

The notation we use for the antiderivative of f is,

$$\int f(x) \mathrm{d}x$$

NB: The "$\mathrm{d}x$" is <u>not</u> optional, just as the "$\mathrm{d}x$" in $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not optional!

The symbol "$\int$" is an elongated/stylized "s". It stands for "sum", but we won't show why until later.

Terminology:

The operation $\frac{\mathrm{d}}{\mathrm{d}x} f$ differentiates f. The function f' is the derivative of f.

The operation $\int f(x) \mathrm{d}x$ antidifferentiates f. The function $\int f(x) \mathrm{d}x$ is the antiderivative of f.

The operation $\int f(x) \mathrm{d}x$ integrates f. The function $\int f(x) \mathrm{d}x$ is the integral of f.

More, unmotivated, terminology: for $\int f(x) \mathrm{d}x$, the function $f(x)$ is the **integrand**, and the resulting antiderivative is called the **indefinite integral**.

The power rule, redux

Since we know how to take derivatives of certain functions, we also know how to take antiderivatives of certain functions:

$$\frac{\mathrm{d}}{\mathrm{d}x} x^2 = 2x \quad \Longleftrightarrow \quad \int 2x \mathrm{d}x = x^2 + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^5 = 5x^4 \quad \Longleftrightarrow \quad \int 5x^4 \mathrm{d}x = x^5 + c$$

The power rule, redux

Since we know how to take derivatives of certain functions, we also know how to take antiderivatives of certain functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x \iff \int 2x\mathrm{d}x = x^2 + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x}x^5 = 5x^4 \iff \int 5x^4\mathrm{d}x = x^5 + c$$

Hence, we have a *power rule* for integrals.

Theorem (Power rule)

If r is any rational number except -1, then

$$\int x^r \mathrm{d}x = \frac{x^{r+1}}{r+1} + c$$

NB: $r = 0$ is allowed. $r = -1$ is not allowed. (The power rule for derivatives never yields x^{-1} as a derivative.)

Trigonometric functions and linearity

Since we know derivatives of the $\sin$ and $\cos$ functions, we also know corresponding antiderivatives.

Theorem

We have $\int \sin x \mathrm{d}x = -\cos x + c$, *and* $\int \cos x \mathrm{d}x = \sin x + c$.

$\frac{d}{dx} \cos x = -\sin x$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx}(-\cos x) = \sin x$

Trigonometric functions and linearity

Since we know derivatives of the $\sin$ and $\cos$ functions, we also know corresponding antiderivatives.

Theorem

We have $\int \sin x \mathrm{d}x = -\cos x + c$, *and* $\int \cos x \mathrm{d}x = \sin x + c$.

We also know that the derivative is linear, e.g., the derivative of a sum is the sum of derivatives. As a result, antidifferentiation is also a linear operation.

Theorem

Suppose f *and* g *have antiderivatives* F *and* G, *respectively. Then for any constants* c_1 *and* c_2,

$$\int (c_1 f(x) + c_2 g(x))\, \mathrm{d}x = c_1 F(x) + c_2 G(x).$$

F is the general antiderivative of f

$\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$

$\int f(x)g(x)\,dx \neq \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right)$

Example

Using linearity of the integral, evaluate

$$\underbrace{\int (4x + 3x^7)\,\mathrm{d}x}_{(A)}, \quad \underbrace{\int \left(\frac{1}{t^3} - \sqrt[3]{t}\right)\mathrm{d}t}_{(B)}, \quad \underbrace{\int (u^2 - 4\sin u)\,\mathrm{d}u}_{(C)}$$

(A): $\int (4x+3x^7)\,dx = \int 4x\,dx + \int 3x^7\,dx$

$= 4\int x\,dx + 3\int x^7\,dx$

$= 4\cdot\left[\frac{x^2}{1+1} + C\right] + 3\left[\frac{x^{7+1}}{7+1} + k\right]$

$$= 2x^2 + \frac{3}{8}x^8 + \underbrace{(4c + 3k)}_{c_1\text{, arbitrary constant}}, \quad c, k \text{ arbitrary constants.}$$

$$= 2x^2 + \frac{3}{8}x^8 + c_1$$

(B) ✗ $\int (t^3 - \sqrt[3]{t})\,dt = \int (t^3 - t^{1/3})\,dt$

$$= \int t^3\,dt - \int t^{1/3}\,dt$$

$$= \frac{t^4}{4} - \frac{t^{4/3}}{4/3} + c = \frac{t^4}{4} + \frac{3}{4}t^{4/3} + c$$

(B): $\int \left(\frac{1}{t^3} - \sqrt[3]{t}\right)dt = \int (t^{-3} - t^{1/3})\,dt$

$$= \frac{t^{-2}}{-2} - \frac{t^{4/3}}{4/3} + c = -\frac{1}{2t^2} - \frac{3t^{4/3}}{4} + c$$

(C): $\int (u^2 - 4\sin u)\,du = \frac{u^3}{3} + 4\cos u + c$

A more general power rule, I

Here's a trick that we can use to evaluate some relatively complicated integrals:

Note that by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)^{20} = 40x\left(x^2+1\right)^{19}.$$

$= 20\,(x^2+1)^{19} \cdot 2x$

A more general power rule, I

Here's a trick that we can use to evaluate some relatively complicated integrals:

Note that by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)^{20} = 40x\left(x^2+1\right)^{19}.$$

Then by definition of the indefinite integral, we have,

$$\int 40x\left(\underbrace{x^2+1}_{g(x)}\right)^{19}\mathrm{d}x = (x^2+1)^{20} + c.$$

$$\int 40x\,(x^2+1)^{19}\,dx = \int 20\cdot \underbrace{2x}_{g'(x)}\,(g(x))^{19}\,dx = \int 20\,g'(x)\,(g(x))^{19}\,dx$$

$$= [g(x)]^{20} + c$$

A more general power rule, I

Here's a trick that we can use to evaluate some relatively complicated integrals:

Note that by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)^{20} = 40x\left(x^2+1\right)^{19}.$$

Then by definition of the indefinite integral, we have,

$$\int 40x\left(x^2+1\right)^{19}\mathrm{d}x = (x^2+1)^{20} + c.$$

We can generalize this idea with the chain and power rules: suppose $g(x)$ is some differentiable function, and r is any rational number. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x)^r = rg(x)^{r-1}g'(x) \quad \Leftrightarrow \quad \int g(x)^{r-1}g'(x)\mathrm{d}x = \frac{1}{r}g(x)^r + c,$$

where $r \neq 0$ for the second expression.

A more general power rule, II

We can formally state this, replacing r with $r+1$:

Theorem ("Generalized" power rule)

Suppose g is a differentiable function and $r \neq -1$ is a rational number. Then

$$\int g(x)^r g'(x) \mathrm{d}x = \frac{g(x)^{r+1}}{r+1} + c.$$

A more general power rule, II

We can formally state this, replacing r with $r+1$:

Theorem ("Generalized" power rule)

Suppose g is a differentiable function and $r \neq -1$ is a rational number. Then

$$\int g(x)^r g'(x) \mathrm{d}x = \frac{g(x)^{r+1}}{r+1} + c.$$

This rule requires some practice and comfort with derivatives to apply: one must be able to identify $g^r(x)$ and $g'(x)$ as expressions in the integrand.

Example

Evaluate the following expressions:

$$\underbrace{\int 3x^2\left(x^3+3\right)^{45}\mathrm{d}x}_{(A)}, \qquad \int x\left(5x^2+13\right)^{13}\mathrm{d}x, \qquad \underbrace{\int \sin^7 x\ \cos x \mathrm{d}x}_{(C)}$$

(A): $\int \underbrace{3x^2}_{g'(x)}(\underbrace{x^3+3}_{g(x)})^{45}dx \overset{r=45}{=} \int g'(x)\,[g(x)]^{45}dx$

$= \dfrac{(g(x))^{46}}{46} + C$

(c) $\int \sin^7 x \cdot \cos x \, dx$

$$\overset{g(x)=\sin x}{\underset{r=7}{=\!=}} \int (g(x))^7 g'(x) \, dx = \frac{g(x)^8}{8} + C$$

$$= \frac{(\sin x)^{\cancel{7}\,8}}{8} + C$$

Example

Things can be kind of tricky. Evaluate:

$$\int \frac{(\sqrt{x}+3)^{17}}{\sqrt{x}} \mathrm{d}x, \qquad \int \left(x^2+2\right)\left(x^3+6x\right)^5 \mathrm{d}x$$

(A)

(A): $g(x) = \sqrt{x} + 3$, $g'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

$$\int \frac{(\sqrt{x}+3)^{17}}{\sqrt{x}} dx = \int 2 \frac{(\sqrt{x}+3)^{17}}{2\sqrt{x}} dx = 2 \int (g(x))^{17} \cdot g'(x) dx$$

$$= \frac{1}{9} g(x)^{18} + C = \frac{1}{9} (\sqrt{x}+3)^{18} + C$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.