Math 1210: Calculus I
Antiderivatives

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 3.8

“Undoing” differentiation

We’ve been discussing the task of differentiating or taking derivatives:

$$f(x) \xrightarrow{\frac{\mathrm{d}}{\mathrm{d}x}} f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}f}{\mathrm{d}x},$$

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The task of inverting differentiation is called **antidifferentiation** or **integration**.

The function f is the **antiderivative** of f'.

A first observation

Recall: If two functions f and g are differentiable, then

$$f'(x) = g'(x) \quad \text{if and only if} \quad f(x) = g(x) + c,$$

for some constant c.

An immediate corollary: If $f(x)$ is an(y) antiderivative of f, then $f(x) + c$ is an(other) antiderivative of f.

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Example (Example 3.8.1)

Find an antiderivative of $f(x) = 4x^3$ on $(-\infty, \infty)$.
(Ans: $F(x) = x^4$, or $F(x) = x^4 + 1$, or ...)

“General” antiderivatives

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In addition, if f is an antiderivative of f', then the only possible antiderivatives of f' are of the form $f(x) + c$ for some constant c.

For this reason, if $f(x)$ is a(ny) antiderivative of $f'(x)$, then we call $f(x) + c$ for arbitrary constant c the **general antiderivative** of f'.

"General" is often omitted, and we simply say *the* antiderivative when referring to the general antiderivative.

Antiderivative notation

The notation we use for the derivative of f is f', or $\frac{\mathrm{d}}{\mathrm{d}x} f(x)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}$.

The notation we use for the antiderivative of f is,

$$\int f(x) \mathrm{d}x$$

NB: The "$\mathrm{d}x$" is <u>not</u> optional, just as the "$\mathrm{d}x$" in $\frac{\mathrm{d}f}{\mathrm{d}x}$ is not optional!

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Terminology:

The operation $\frac{\mathrm{d}}{\mathrm{d}x} f$ differentiates f. The function f' is the derivative of f.

The operation $\int f(x) \mathrm{d}x$ antidifferentiates f. The function $\int f(x) \mathrm{d}x$ is the antiderivative of f.

The operation $\int f(x) \mathrm{d}x$ integrates f. The function $\int f(x) \mathrm{d}x$ is the integral of f.

More, unmotivated, terminology: for $\int f(x) \mathrm{d}x$, the function $f(x)$ is the **integrand**, and the resulting antiderivative is called the **indefinite integral**.

The power rule, redux

Since we know how to take derivatives of certain functions, we also know how to take antiderivatives of certain functions:

$$\frac{\mathrm{d}}{\mathrm{d}x} x^2 = 2x \quad \Longleftrightarrow \quad \int 2x \mathrm{d}x = x^2 + c$$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^5 = 5x^4 \quad \Longleftrightarrow \quad \int 5x^4 \mathrm{d}x = x^5 + c$$

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Hence, we have a *power rule* for integrals.

Theorem (Power rule)

If r is any rational number except -1, then

$$\int x^r \mathrm{d}x = \frac{x^{r+1}}{r+1} + c$$

NB: $r = 0$ is allowed. $r = -1$ is not allowed. (The power rule for derivatives never yields x^{-1} as a derivative.)

Trigonometric functions and linearity

Since we know derivatives of the $\sin$ and $\cos$ functions, we also know corresponding antiderivatives.

Theorem

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We also know that the derivative is linear, e.g., the derivative of a sum is the sum of derivatives. As a result, antidifferentiation is also a linear operation.

Theorem

Suppose f *and* g *have antiderivatives* F *and* G*, respectively. Then for any constants* c_1 *and* c_2,

$$\int (c_1 f(x) + c_2 g(x)) \, \mathrm{d}x = c_1 F(x) + c_2 G(x).$$

Examples

Example

Using linearity of the integral, evaluate

$$\int \left(4x + 3x^7\right) \mathrm{d}x, \quad \int \left(\frac{1}{t^3} - \sqrt[3]{t}\right) \mathrm{d}t, \quad \int \left(u^2 - 4\sin u\right) \mathrm{d}u$$

A more general power rule, I

Here's a trick that we can use to evaluate some relatively complicated integrals:

Note that by the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right)^{20} = 40x\left(x^2+1\right)^{19}.$$

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We can generalize this idea with the chain and power rules: suppose $g(x)$ is some differentiable function, and r is any rational number. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}g(x)^r = rg(x)^{r-1}g'(x) \quad \Leftrightarrow \quad \int g(x)^{r-1}g'(x)\mathrm{d}x = \frac{1}{r}g(x)^r + c,$$

where $r \neq 0$ for the second expression.

A more general power rule, II

We can formally state this, replacing r with $r+1$:

Theorem ("Generalized" power rule)

Suppose g is a differentiable function and $r \neq -1$ is a rational number. Then

$$\int g(x)^r g'(x) \mathrm{d}x = \frac{g(x)^{r+1}}{r+1} + c.$$

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This rule requires some practice and comfort with derivatives to apply: one must be able to identify $g^r(x)$ and $g'(x)$ as expressions in the integrand.

Examples

Example

Evaluate the following expressions:

$$\int 3x^2\left(x^3+3\right)^{45}\mathrm{d}x, \qquad \int x\left(5x^2+13\right)^{13}\mathrm{d}x, \qquad \int \sin^7 x\ \cos x \mathrm{d}x$$

Example

Things can be kind of tricky. Evaluate:

$$\int \frac{(\sqrt{x}+3)^{17}}{\sqrt{x}} \mathrm{d}x, \qquad \int \left(x^2+2\right)\left(x^3+6x\right)^5 \mathrm{d}x$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.