Math 1210: Calculus I
Introduction to area

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.1

Polygonal area

You probably know several formulas for computing planar areas.

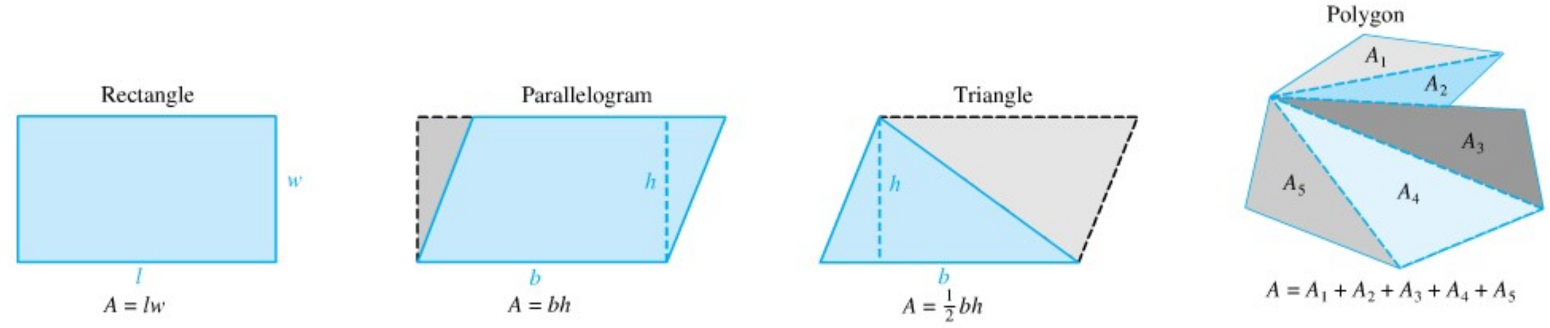


Figure 1

While we can compute areas of quite general polygons, there are only a few other types of shapes for which we know area formulas. (E.g., circles)

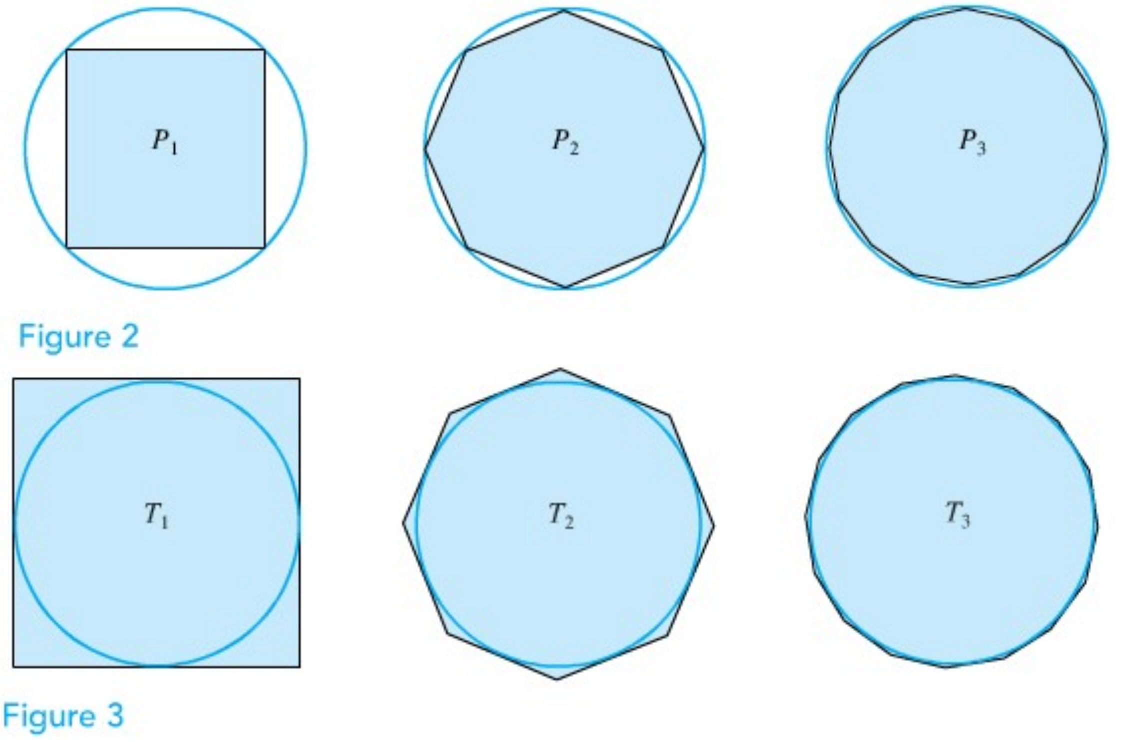
Area

In generalizing area to a general principle for, say, curved regions, we should keep some desiderata in mind.

1. Area is *non-negative*
2. Area is *monotonic*: if a region A_1 is a subset of A_2, then it must have smaller area than A_2.
3. Area is invariant to rigid rotations and translations. (*Congruent* regions have identical area.)
4. Area is additive: two non-overlapping regions together have area equal to the sum of the areas of each region.
5. Area should be consistent with the familiar formula for area of rectangles.

Polygonal area as an approximation

The previous requirements lead us to the following idea.



Let's approximate curved regions with polygons, say a polygon with n sides.
(Top: "**Inscribed**" polygons. Bottom: "**Circumscribed**" polygons.)

Polygonal area as an approximation

We'll say that the area of the n-sided polygon *approximates* the area of the curved region.

The actual area will be the result of taking the limit as $n \uparrow \infty$, in such a way that the polygonal boundary converges to the curved region's boundary.

Since polygonal areas are sums of areas of simple regions (triangles, rectangles), it seems we'll need notation for, say, the sum of n areas.

Summation notation: Σ

There is a standard notation for taking sums of numbers in mathematics: Σ ("sigma") notation.

Suppose I ask you to sum up all the integers from 1 to 100. We'd write this as,

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The above is rather laborious to write. And annoying to manipulate. (How would you write π times the above summation?) $\pi(1+2+3+\ldots+99+100)$

Summation notation is meant to make things easier to write. For example, we write the above as,

$$1 + 2 + 3 + \ldots + 99 + 100 = \sum_{n=1}^{100} n.$$

More explicitly, this is "Sum n for $n = 1$ to $n = 100$". (Greek Σ corresponds to "S".)

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upper limit

$$1 + 2 + 3 + \ldots + 99 + 100 = \sum_{n=1}^{100} n.$$

argument

index

lower limit

More explicitly, this is "Sum n for $n = 1$ to $n = 100$". (Greek Σ corresponds to "S".)

It's worth emphasizing some terminology:

- The n in the subscript: the **summation index**
- The number 1: the **lower limit** (of the index)
- The number 100: the **upper limit** (of the index)
- The expression n after $\sum$: the **argument** (being summed)

Summation notation: dummy indices

One can use summation notation to write all sorts of strange and general expressions:

$$\sum_{i=3}^{14} i^2, \qquad \sum_{q=3}^{100} (3q-3), \qquad \sum_{r=0}^{3} 2,$$

$r=0 \quad r=1 \quad r=2 \quad r=3$

$2+2+2+2=8$

"sum of i^2 as i goes from 3 to 14"

$(3)^2+(4)^2+(5)^2+\cdots+(14)^2$

$9+16+25+\cdots+196$

$(3\cdot 3-3)+(3\cdot 4-3)+\cdots+(3\cdot 100-3)$

$6+9+\cdots+297$

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One can use summation notation to write all sorts of strange and general expressions:

$$\sum_{i=3}^{14} i^2, \quad \sum_{q=3}^{100} (3q-3), \quad \sum_{r=0}^{3} 2,$$

One observation: the choice of letter for the summation index is *irrelevant*. It's a **dummy index**, i.e., the choice of letter doesn't matter.

In particular, <u>all</u> the following mean exactly the same thing:

$$\sum_{j=1}^{50} (j^2 - j), = \sum_{x=1}^{50} (x^2 - x), = \sum_{\smiley=1}^{50} (\smiley^2 - \smiley).$$

Obviously, some choices of summation index symbol are more confusing than others.

Summation notation: sequences

We can also write summation arguments with some additional shorthand. Consider the sum,

$$\sum_{n=1}^{30} (3n^2 - 4n + 2).$$

We can define a **sequence** of numbers,

$$a_n = 3n^2 - 4n + 2.$$

For example, $a_{13} = 3(13)^2 - 4(13) + 2 = 457$.

$$\sum_{n=1}^{100} (3n^2 - 4n + 2) = \sum_{n=1}^{100} a_n, \quad a_n = 3n^2 - 4n + 2$$

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Then we can equivalently write the sum above as,

$$\sum_{n=1}^{30} (3n^2 - 4n + 2) = \sum_{n=1}^{30} a_n, \qquad \text{where } a_n = 3n^2 - 4n + 2.$$

This shorthand can be quite useful in many situations.

(Like before, the choice of n as the summation index symbol is arbitrary and irrelevant: any other symbol could be chosen.)

Properties of summation

Through some straightforward analysis (and understanding of addition), the Σ operation has some, perhaps unsurprising, properties.

Theorem (Linearity of summation)

Let a_n *and* b_n *be any two sequences. Then*

- $\displaystyle\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
- $\displaystyle\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$ *for any constant* c.

$\left(\sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right.$ AND

$\left. \sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i \right)$

E.g.: $\displaystyle\sum_{r=0}^{3} 2 = 2\sum_{r=0}^{3} 1 = 2(1+1+1+1) = 8$

Example

Suppose that $\sum_{n=1}^{40} q_n = 30$ and $\sum_{j=1}^{40} y_j = -3$. Compute

$$\sum_{i=1}^{40} (3q_i - 4y_i + 2)$$

$$= \sum_{i=1}^{40} 3q_i - \overset{4}{\sum_{i=1}^{40}} y_i + \sum_{i=1}^{40} 2$$

$$= 3\left(\sum_{i=1}^{40} q_i\right) - 4\left(\sum_{i=1}^{40} y_i\right) + \sum_{i=1}^{40} 2 = 3(30) - 4(-3) + 2 \cdot 40$$

$$= 90 + 12 + 80 = 182$$

Examples

Example (Example 4.2.2)

Show that

$$\sum_{i=1}^{n}(a_{i+1}-a_i) = a_{n+1}-a_1, \qquad \sum_{i=1}^{n}\left[(i+1)^2 - i^2\right] = (n+1)^2 - 1.$$

$(a_i = i^2)$

$\parallel$

$(\cancel{a_2} - \underline{\underline{a_1}}) + (\cancel{a_3} - \cancel{a_2}) + (a_4 - \cancel{a_3}) + \dots + (\cancel{a_n} - a_{n-1}) + (\underline{\underline{a_{n+1}}} - \cancel{a_n})$

$= a_{n+1} - a_1$

("Telescoping sums")

Some useful sums

There are a couple of explicit sums that are useful enough to point out:

- $\displaystyle\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$ (E.g. $\displaystyle\sum_{i=1}^{100} i = 1+2+\cdots+99+100 = \frac{1}{2}(100)(101)$)
- $\displaystyle\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$

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There are a couple of explicit sums that are useful enough to point out:

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Example

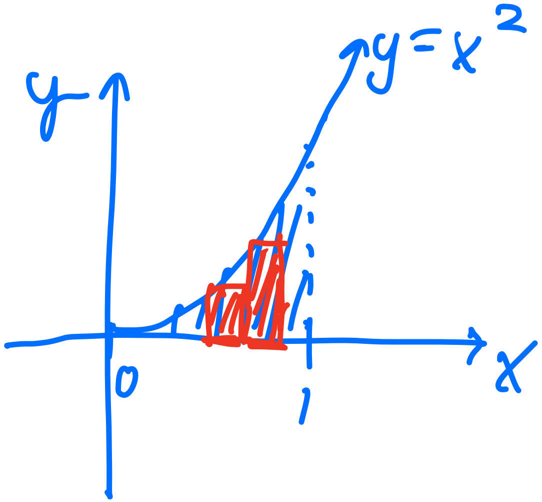
Compute a formula for $\displaystyle\sum_{j=1}^{n}(j-1)(j+4)$. (a number that depends on n)

$$= \sum_{j=1}^{n}\left(j^2 + 3j - 4\right) = \sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j - \sum_{j=1}^{n} 4$$

$$= \frac{1}{6}n(n+1)(2n+1) + \frac{3}{2}n(n+1) - 4n$$

Back to the main point: areas via polygons, I

We now have some ammunition to tackle some nontrivial problems. Here's one to start with: What's the area between the curves of $y = 0$ and $y = x^2$ for x inside the interval $[0, 1]$?
(We often call this the "area under the curve of $y = x^2$" between $x = 0$ and $x = 1$.)



area = ?

idea: approximate this area w/ a polygon.

One easy choice: rectangles

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Let's approximate this with the area of an inscribed polygon:

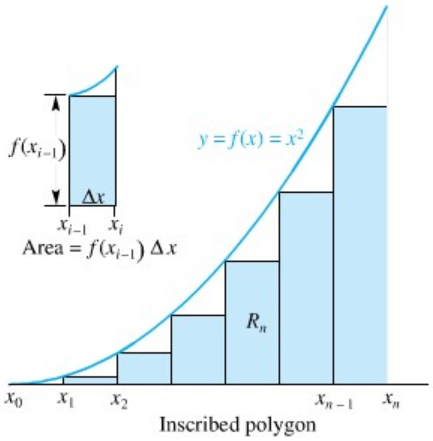
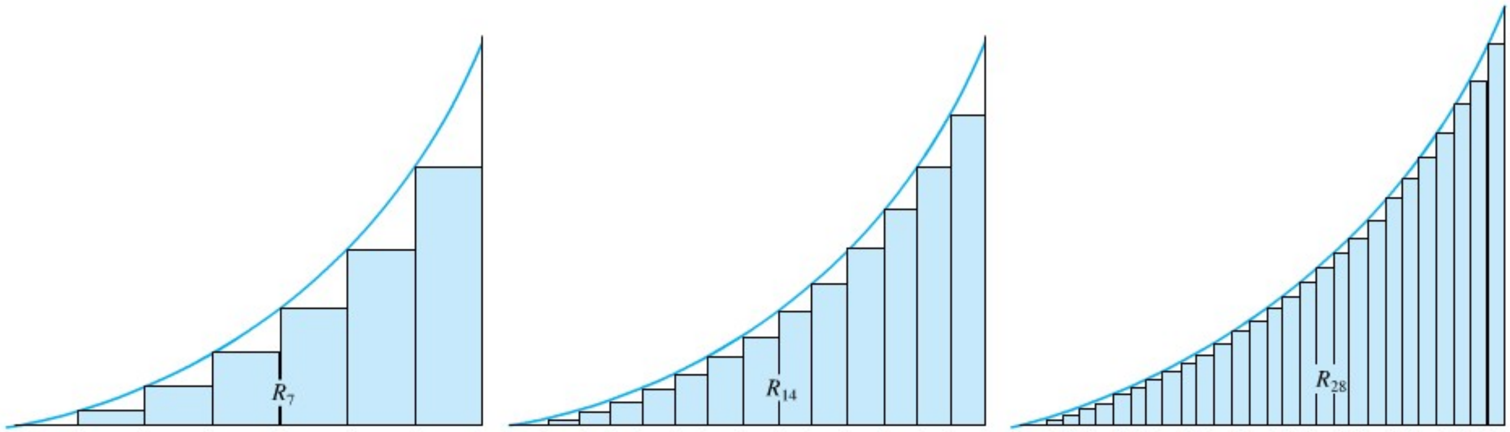


Figure 7

Here, n denotes the number of rectangles we use, and R_n is the corresponding polygon (just formed by taking the collection of rectangles depicted together.)

Back to the main point: areas via polygons, II

Our goal will be to take $n \uparrow \infty$ to compute the area:



The core computation, I

We now formally construct our summation:

We partition the interval $[0,1]$ into n rectangles of equal width. To do this we construct the gridpoints,

$$x_0 = 0, \qquad x_1 = \frac{1}{n}, \qquad x_2 = \frac{2}{n} \qquad \cdots \qquad x_n = \frac{n}{n} = 1.$$

In particular, we introduce the rectangle width Δx: $x_j - x_{j-1} = \frac{1}{n} = \Delta x$.

height is $y=x^2$ evaluated at $x=x_2$

$\left(\left(2/n\right)^2\right)$

Shown: "rectangle 3"

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In particular, we introduce the rectangle width Δx: $x_j - x_{j-1} = \frac{1}{n} = \Delta x$. To form the inscribed polygon, we assign a height equal to f evaluated at the left-hand point of the rectangle base (since f is increasing).

$$\begin{aligned}
&\text{Rectangle 1: width } \Delta x, \text{ height } f(x_0) = 0 \\
&\text{Rectangle 2: width } \Delta x, \text{ height } f(x_1) = \left(\frac{1}{n}\right)^2 \\
&\vdots \\
&\text{Rectangle n: width } \Delta x, \text{ height } f(x_{n-1}) = \left(\frac{n-1}{n}\right)^2
\end{aligned}$$

The core computation, II

The area of the polygon is the sum of all the areas:

$$\text{Area of } R_n = \underbrace{\Delta x f(x_0)}_{\text{area of rect 1}} + \underbrace{\Delta x f(x_1)}_{\text{area of rect 2}} + \ldots + \underbrace{\Delta x f(x_{n-1})}_{\text{area of rect. } n} = \sum_{j=1}^{n} \Delta x f(x_{j-1}) = \sum_{j=1}^{n} \frac{1}{n}\left(\frac{(j-1)}{n}\right)^2$$

$f(x) = x^2$

$x_{j-1} = \frac{j-1}{n}$

(NB: $f(x_0) = 0$)

Area of R_n: $\sum_{j=1}^{n} \frac{1}{n}\left(\frac{j-1}{n}\right)^2 = \frac{1}{n^3}\sum_{j=1}^{n}(j-1)^2 = \frac{1}{n^3}\left[\sum_{j=1}^{n} j^2 - 2\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1\right]$

Recall: $\sum_{j=1}^{n} j = \frac{1}{2}n(n+1)$, $\sum_{j=1}^{n} j^2 = \frac{1}{6}n(n+1)(2n+1)$

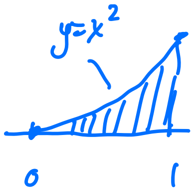
$$\text{Area of } R_n = \frac{1}{n^3}\left[\frac{1}{6}n(n+1)(2n+1) - 2\,\frac{1}{2}\,n(n+1) + n\right]$$

$$= \frac{1}{n^3}\left[\frac{1}{6}(\overset{2}{n^3} + 3n^2 + n) - n^2 - \cancel{n} + \cancel{n}\right]$$

$$+3n^2 \qquad\qquad -\frac{1}{6}\cdot 6n^2$$

$$= \frac{1}{6n^3}\left[2n^3 - 3n^2 + n\right] = \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

$$\lim_{n\to\infty}(\text{Area of } R_n) = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^3}\right) = \frac{1}{3}$$



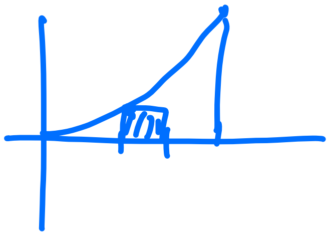
The summary

We've confirmed the following: the area under the curve of $y = x^2$ on the interval $[0, 1]$ is given by,

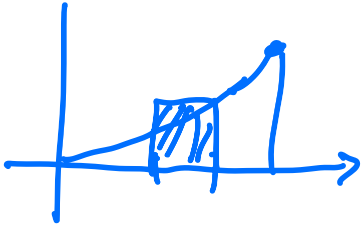
$$\text{Area} = \lim_{n\to\infty}\left(\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

(This is, in fact, the actual area!)

what about



vs



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You may be concerned that our choice of *inscribed* polygons has affected the answer.
What if we do the same exercise with circumscribed polygons? The answer will be the same.

Try it: All that changes is now the area of polygon R_n is $\sum_{j=1}^{n} \Delta x f(x_j)$.

vs $f(x_{j-1})$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.