Math 1210: Calculus I
The definite integral

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.2

Area through polygonal sums

We have seen that area for curved regions can be computed by

1. approximating the region by a polygon formed from n rectangles
2. taking $n \uparrow \infty$ in a limit

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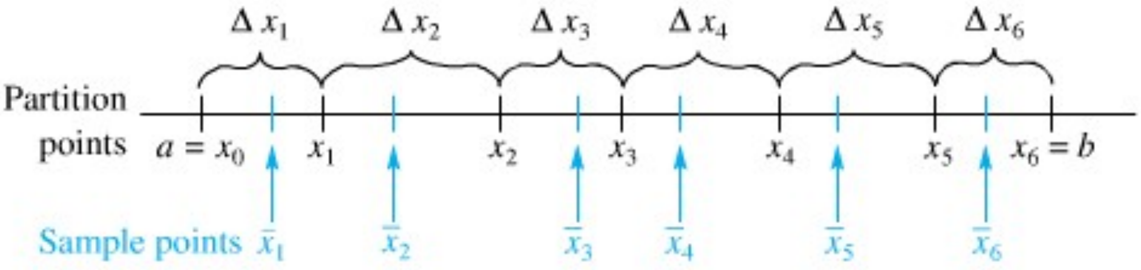
For "most" functions of our interest, the choice of which point we evaluate the function at won't matter.

Area is
$$\sum_{j=1}^{n} (\Delta x_j)\, f(\bar{x}_j)$$



A Partition of $[a, b]$ with Sample Points $\bar{x}_i$

For convenience, over the jth subinterval $[x_{j-1}, x_j]$, we let $\bar{x}_j$ denote some *sample point* inside this intervalthat we use to construct the rectangle.

Riemann sums

With a *partition*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

of the interval $[a, b]$ into n subintervals, then, + making a choice for $\bar{x}_j$ in $[x_{j-1}, x_j]$,

$$R = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i, \qquad \Delta x_i := x_i - x_{i-1},$$

is the **Riemann sum** for the function f corresponding to this partition.

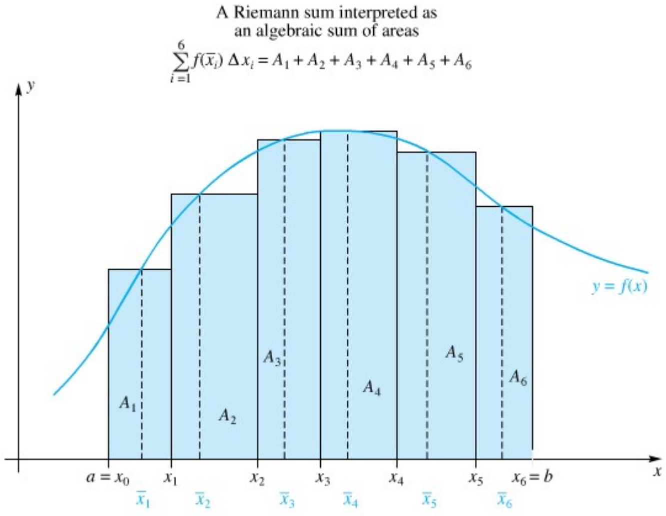


Figure 3

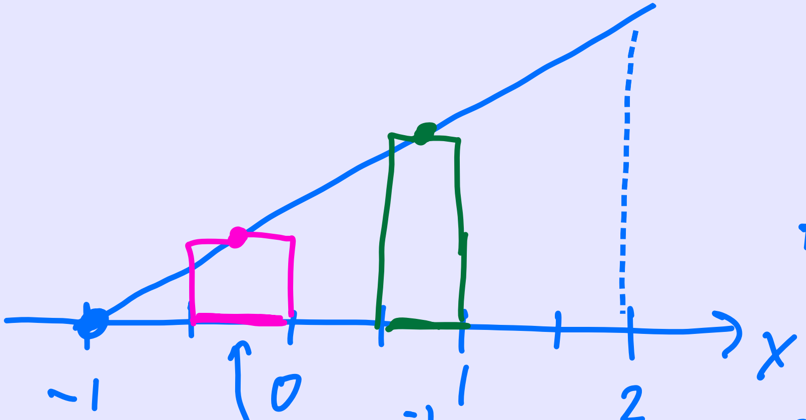
Riemann sum example

Example

Evaluate the Riemann sum for $f(x) = x + 1$ on the interval $[-1, 2]$ using the equally spaced partition points

$$a = -1 < -0.5 < 0 < 0.5 < 1 < 1.5 < 2, = b$$

with the sample points $\bar{x}_i$ being the midpoint of the ith subinterval.



midpoint is -0.25

Sum of areas of rectangles:

$$(0.5)(0.25) + \underline{(0.5)(0.75)} + (0.5)(1.25)$$

$$+ \underline{(0.5)(1.75)} + (0.5)(2.25) + (0.5)(2.75)$$

$$= 0.5[1 + 3 + 5] = 9/2$$

"Signed area"

One observation from the last example: Riemann sums could be negative!
In particular, this is possible if $f(x) < 0$.
Hence, a Riemann sum is not necessarily computing a non-negative "area", but we could say it's computing a *signed area*.

- The *signed area* under the curve of $y = f(x)$ is positive when $f(x) > 0$.
- The *signed area* under the curve of $y = f(x)$ is negative when $f(x) < 0$.
  (Here, "under the curve" really means between the curve and the horizontal line $y = 0$.)

The limit of Riemann sums

As before, the punchline happens when we take a limit in n:

Definition

Suppose a function f is defined on the interval $[a, b]$. We say that f is **integrable** on $[a, b]$ if,

$$\lim_{\substack{n \to \infty \\ \max_i \Delta x_i \to 0}} \sum_{i=1}^{n} f(\bar{x}_i) \Delta x_i$$

exists for any choice of sample point $\bar{x}_i$ and any limit of partitions.

When this limit exists, we denote,

a number!

$$\lim_{\substack{n \to \infty \\ \max_i \Delta x_i \to 0}} \sum_{i=1}^{n} f(\bar{x}_i) \Delta x_i = \int_a^b f(x) \mathrm{d}x,$$

this is the signed area under the curve of f(x).

and call this quantity the **definite integral** of f over $[a, b]$.

The number,

$$\int_a^b f(x)\mathrm{d}x,$$

can be positive or negative, depending on which parts of the curve $f(x)$ are above or below the x-axis.

$y = f(x)$

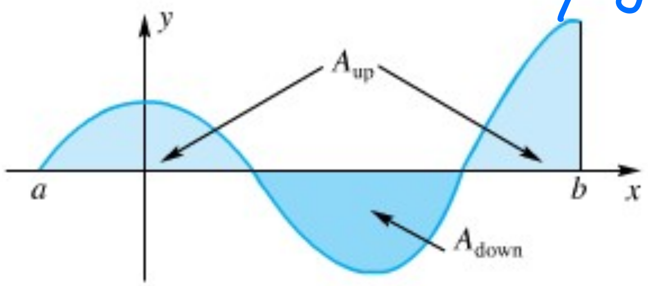


Figure 7

A_{up}, A_{down}: standard planar areas (non-negative)

From the above geometry with $A_{\text{up}}, A_{\text{down}} > 0$, we could write,

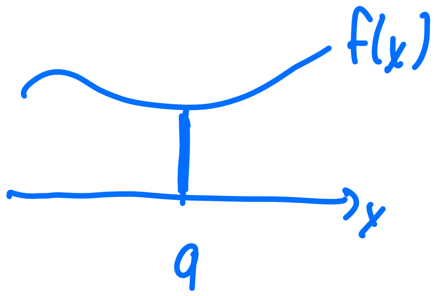
$$\int_a^b f(x)\mathrm{d}x = A_{\text{up}} - A_{\text{down}},$$

could be any real number.

Definite integral: notation and conventions, II

Note that

$$\int_a^a f(x)\mathrm{d}x,$$



is an area corresponding with a line, and should therefore be 0:

$$\int_a^a f(x)\mathrm{d}x = 0.$$

Definite integral: notation and conventions, II

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When $a < b$, we know what $\int_a^b f(x)\mathrm{d}x$ means.
What if $b < a$? By convention, we define this as, E.g. $\int_2^1 x^2 dx = -\int_1^2 x^2 dx$

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x,$$

i.e., swapping the limits of the definite integral multiplies the result by -1.

Definite integral: notation and conventions, III

Finally, we note that the two numbers

$$\int_a^b f(x)\mathrm{d}x, \qquad \int_a^b f(y)\mathrm{d}y,$$

correspond to exactly the same thing.

$y(x)$

$\downarrow$

$dy = y'(x)\,dx$

In particular, the variables x and y above are **dummy variables**.
All the following expressions are the same:

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(g)\mathrm{d}g = \int_a^b f(☺)\mathrm{d}☺$$

This is why it's *very important* to <u>not omit</u> the $\mathrm{d}x$ notation!

Which functions are integrable?

We require the limit of Riemann sums to converge in order for a function to be integrable.

What kinds of functions are integrable?

Theorem

Suppose f is a bounded function on $[a,b]$, and is continuous except at a finite number of points. Then f is integrable on $[a,b]$.

f bounded + continuous everywhere on $[a,b] \Rightarrow f$ is integrable

Which functions are integrable?

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Theorem

Suppose f is a bounded function on $[a, b]$, and is continuous except at a finite number of points. Then f is integrable on $[a, b]$.

Hence, many functions we know are integrable:

- polynomials
- rational functions, away from vertical asymptotes
- sine and cosine functions

Computing definite integrals

D27-S11(a)

Since integrable functions have convergent Riemann sums for any choice of partition, we may choose a(ny) convenient partition.

Example

Evaluate $\int_{-1}^{4}(x+3)\mathrm{d}x$.

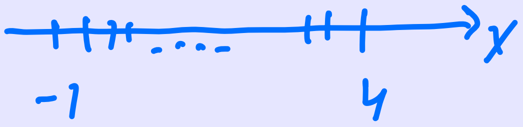
We compute this in essentially the same way as we computed areas before.

$a=-1, \quad b=4$

$x_0=-1$

$x_n=4$

$x_j=-1+j\frac{5}{n}, \quad j=0,1,\ldots,n$



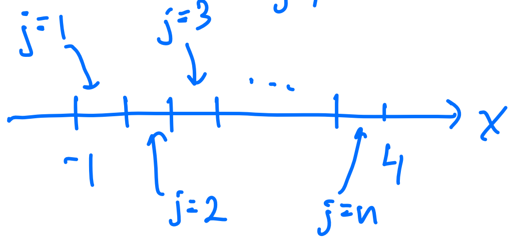
Construct equispaced partition w/ n subintervals

$\Delta x = x_j - x_{j-1} = 5/n$

Choose a sample point in each interval: choose $\bar{x}_j = x_j$ (right-hand point)

$$\Rightarrow f(\bar{x}_j) = f(x_j) = x_j + 3 = -1 + j\frac{5}{n} + 3 = 2 + \frac{5}{n}j$$

Riemann sum: $\sum_{j=1}^{n} f(\bar{x}_j)\Delta x = \sum_{j=1}^{n} \left(2 + \frac{5}{n}j\right)\frac{5}{n}$



Area "for $j=1$": $(x_1 - x_0)\cdot f(\bar{x}_1)$

$= (x_1 - x_0)\, f(x_1)$

Riemann sum: $\sum_{j=1}^{n} \left(2 + \frac{5}{n}j\right)\frac{5}{n} = \frac{5}{n}\sum_{j=1}^{n} 2 + \left(\frac{5}{n}\right)^2 \sum_{j=1}^{n} j$

$$= \frac{5}{n}(2n) + \left(\frac{5}{n}\right)^2 \cdot \frac{n(n+1)}{2}$$

$$= 10 + \frac{25}{2}\,\frac{n^2+n}{n^2}$$

$$\lim_{n\to\infty} (\text{Rieman Sum}) = 10 + \frac{25}{2} = 45/2$$

Adding areas

Consider the following figure. By simple geometry, we can guess the definite integral over $[a, c]$.

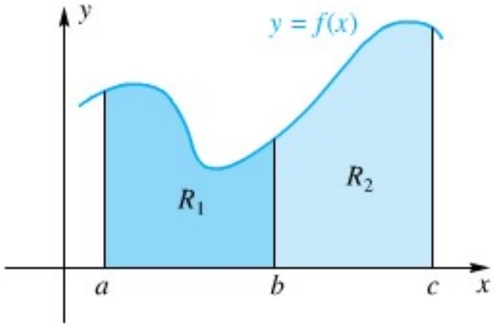


Figure 11

Hence, the following is true:

Theorem

Assume f is integrable on $[a, c]$, and that b is some point in $[a, c]$. Then:

$$\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x.$$

Because this should be true for b in [a, c], then we'll assert that it should be true for <u>all</u> b, even those outside [a, c].

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \quad \text{for } \underline{\text{all}}\ b.$$

Then: $$\underbrace{\int_a^a f(x)\,dx}_{=0} = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx$$

$$\Longrightarrow \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

<u>End of material for midterm exam 3</u>

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.