Math 1210: Calculus I
The definite integral

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.2

Area through polygonal sums

We have seen that area for curved regions can be computed by

1. approximating the region by a polygon formed from n rectangles
2. taking $n \uparrow \infty$ in a limit

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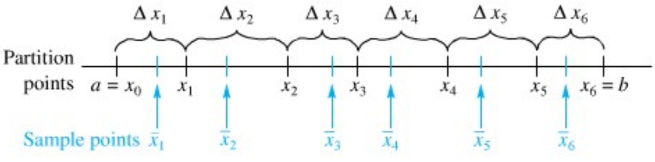
For "most" functions of our interest, the choice of which point we evaluate the function at won't matter.



A Partition of $[a, b]$ with Sample Points $\bar{x}_i$

For convenience, over the jth subinterval $[x_{j-1}, x_j]$, we let $\bar{x}_j$ denote some *sample point* inside this intervalthat we use to construct the rectangle.

Riemann sums

With a *partition*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b,$$

of the interval $[a, b]$ into n subintervals, then,

$$R = \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i, \qquad \Delta x_i := x_i - x_{i-1},$$

is the **Riemann sum** for the function f corresponding to this partition.

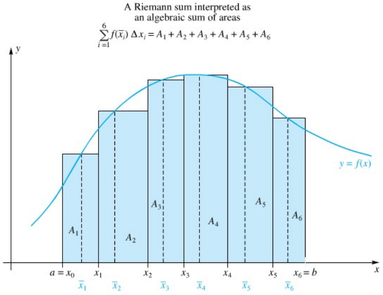


Figure 3

Riemann sum example

Example

Evaluate the Riemann sum for $f(x) = x + 1$ on the interval $[-1, 2]$ using the equally spaced partition points

$$-1 < -0.5 < 0 < 0.5 < 1 < 1.5 < 2,$$

with the sample points $\bar{x}_i$ being the midpoint of the ith subinterval.

"Signed area"

One observation from the last example: Riemann sums could be negative!
In particular, this is possible if $f(x) < 0$.
Hence, a Riemann sum is not necessarily computing a non-negative "area", but we could say it's computing a *signed area*.

- The *signed area* under the curve of $y = f(x)$ is positive when $f(x) > 0$.
- The *signed area* under the curve of $y = f(x)$ is negative when $f(x) < 0$.
  (Here, "under the curve" really means between the curve and the horizontal line $y = 0$.)

The limit of Riemann sums

As before, the punchline happens when we take a limit in n:

Definition

Suppose a function f is defined on the interval $[a, b]$. We say that f is **integrable** on $[a, b]$ if,

$$\lim_{\substack{n \to \infty \\ \max_i \Delta x_i \to 0}} \sum_{i=1}^{n} f(\overline{x}_i) \Delta x_i$$

exists for any choice of sample point $\overline{x}_i$ and any limit of partitions.

When this limit exists, we denote,

$$\lim_{\substack{n \to \infty \\ \max_i \Delta x_i \to 0}} \sum_{i=1}^{n} f(\overline{x}_i) \Delta x_i = \int_a^b f(x) \mathrm{d}x,$$

and call this quantity the **definite integral** of f over $[a, b]$.

Definite integral: notation and conventions, I

The number,

$$\int_a^b f(x)\mathrm{d}x,$$

can be positive or negative, depending on which parts of the curve $f(x)$ are above or below the x-axis.

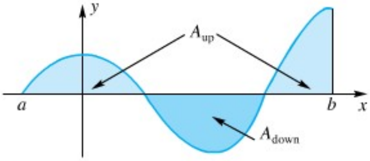


Figure 7

From the above geometry with $A_{\mathrm{up}}, A_{\mathrm{down}} > 0$, we could write,

$$\int_a^b f(x)\mathrm{d}x = A_{\mathrm{up}} - A_{\mathrm{down}},$$

Definite integral: notation and conventions, II

Note that

$$\int_a^a f(x)\mathrm{d}x,$$

is an area corresponding with a line, and should therefore be 0:

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Definite integral: notation and conventions, II

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When $a < b$, we know what $\int_a^b f(x)\mathrm{d}x$ means.
What if $b < a$? By convention, we define this as,

$$\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x,$$

i.e., swapping the limits of the definite integral multiplies the result by -1.

Definite integral: notation and conventions, III

Finally, we note that the two numbers

$$\int_a^b f(x)\mathrm{d}x, \qquad \int_a^b f(y)\mathrm{d}y,$$

correspond to exactly the same thing.

In particular, the variables x and y above are **dummy variables**.
All the following expressions are the same:

$$\int_a^b f(x)\mathrm{d}x = \int_a^b f(g)\mathrm{d}g = \int_a^b f(☺)\mathrm{d}☺$$

This is why it's *very important* to <u>not omit</u> the $\mathrm{d}x$ notation!

Which functions are integrable?

We require the limit of Riemann sums to converge in order for a function to be integrable.

What kinds of functions are integrable?

Theorem

Suppose f is a bounded function on $[a, b]$, and is continuous except at a finite number of points. Then f is integrable on $[a, b]$.

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Theorem

Suppose f is a bounded function on $[a, b]$, and is continuous except at a finite number of points. Then f is integrable on $[a, b]$.

Hence, many functions we know are integrable:

- polynomials
- rational functions, away from vertical asymptotes
- sine and cosine functions

Computing definite integrals

Since integrable functions have convergent Riemann sums for any choice of partition, we may choose a(ny) convenient partition.

Example

Evaluate $\int_{-1}^{4} (x+3)\mathrm{d}x$.

Adding areas

Consider the following figure. By simple geometry, we can guess the definite integral over $[a, c]$.

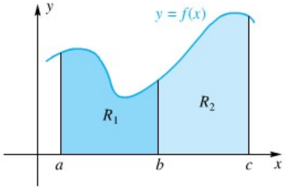

Figure 11

Hence, the following is true:

Theorem

Assume f is integrable on $[a, c]$, and that b is some point in $[a, c]$. Then:

$$\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x.$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.