Math 1210: Calculus I
The first Fundamental Theorem of Calculus

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.3

The Fundamental Theorem of Calculus

The study of areas appears to be unrelated to calculus (derivatives).

One of the main punchlines in all of calculus is that derivatives (slopes of tangent lines) and areas are intimately related.

This relation is the Fundamental Theorem of Calculus.

We will study one "piece" of this relation, called the "First" Fundamental Theorem of Calculus.

Accumulation Functions

Revealing the connection between areas and derivatives requires us to construct a new entity: an *accumulation* function.

This is a function that measures the cumulative area under the curve of a function.

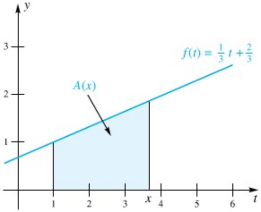

Figure 1

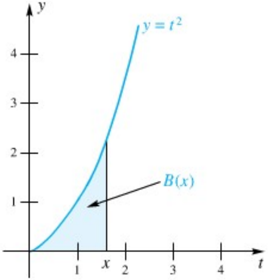

Figure 2

These functions explicitly tell us about (accumulation of) area under a curve.

Explicit accumulation functions

Formally: given some function $f(t)$ and a "starting point" $t = a$, the **accumulation function** $A(x)$ is the area under the curve of $f(t)$ from $t = a$ to $t = x$:

$$A(x) = \int_a^x f(t)\mathrm{d}t$$

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Example

Compute the accumulation function $A(x)$ for $f(t) = t^2$ with $t \geqslant 0$.

A curious observation

We now have an example of a particular accumulation function, $A(x) = x^3/3$ for the area under the curve of $f(t) = t^2$ over t in $[0, x]$.

$$A(x) = \frac{x^3}{3} = \int_0^x t^2 \mathrm{d}t.$$

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For curiosity's sake, we can take a derivative:

$$A'(x) = x^2 = f(x)$$

This last equality is the interesting part: the derivative of $A(x)$ equals the function f that we started with.

That this is true in general is the First Fundamental Theorem of Calculus.

A motivation for the FTC

Why should $A'(x) = f(x)$? Let's use the definition:

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

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And so:

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The First Fundamental Theorem of Calculus

Theorem (First FTC)

Let f be continuous on $[a, b]$, and let x be a(ny) point in (a, b). Then:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\mathrm{d}t = f(x).$$

I.e., if $A(x) = \int_a^x f(t)\mathrm{d}t$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} A(x) = f(x)$$

A more formal argument for an FTC proof

Recall that we have almost already proved the FTC. We know that:

$$A'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \mathrm{d}t.$$

What we need to do is justify that the limit equals $f(x)$.

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What we need to do is justify that the limit equals $f(x)$.

Here's a useful tool for accomplishing this:

Theorem (Comparison Property)

Assume f and g are integrable on $[a, b]$, and that $f(x) \leqslant g(x)$ for all x in $[a, b]$. Then,

$$\int_a^b f(x) \mathrm{d}x \leqslant \int_a^b g(x) \mathrm{d}x$$

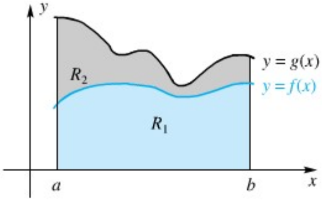

Figure 5

Sandwiching the definite integral, I

We can now be more formal about approximating our integral: First, define

$$m_h : \text{The minimum value of } f \text{ over the interval } [x, x+h].$$
$$M_h : \text{The maximum value of } f \text{ over the interval } [x, x+h].$$

Since f is continuous, these values exist.

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Since f is continuous, these values exist.

In particular, it must be true that

$$\lim_{h \to 0} m_h = f(x)$$
$$\lim_{h \to 0} M_h = f(x)$$

(E.g., assuming these limits exist, they could not possibly converge to any other value.)

Sandwiching the definite integral, II

Now, using the sandwich theorem for limits,

$$\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \leqslant \frac{1}{h}\int_x^{x+h} M_h \mathrm{d}t = M_h,$$

and similarly for a lower bound.

Sandwiching the definite integral, II

Now, using the sandwich theorem for limits,

$$\frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \leqslant \frac{1}{h}\int_x^{x+h} M_h \mathrm{d}t = M_h,$$

and similarly for a lower bound.

Therefore,

$$m_h \leqslant \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t \leqslant M_h,$$

and by the Sandwich Theorem, $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\mathrm{d}t = f(x)$.

Linearity of the definite integral

One more fundamental property of the definite integral is linearity.
(The definite integral is comprised of sums and limits, which are both linear, so linearity of the definite integral is perhaps not surprising.)

Theorem

Let f and g be integrable on $[a, b]$, and let c_1 and c_2 be constants. Then $c_1 f + c_2 g$ is an integrable function, and

$$\int_a^b (c_1 f(x) + c_2 g(x))\,\mathrm{d}x = c_1 \int_a^b f(x)\mathrm{d}x + c_2 \int_a^b g(x)\mathrm{d}x$$

Examples

Example

Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-3}^{x} t^5 \mathrm{d}t\right]$$

Example

Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_{-3}^{x} \frac{\sin u^{3/2}}{\sqrt{u^8+1}}\mathrm{d}u\right]$$

Example

Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_x^4 (t^7 + t)^{1/3}\mathrm{d}t\right]$$

Example

Compute

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^{x^2} t^3 \mathrm{d}t\right]$$

Example

Let $v(t)$ be the one-dimensional velocity of an object at time t. Show that the accumulation function,

$$A(t) = \int_a^t v(s)\mathrm{d}s,$$

is the position of the object relative to its location at $t = a$.

Examples

Example

Evaluate

$$\int_0^{\pi} \sin x \mathrm{d}x$$

Example

Show that if $a(x)$ and $b(x)$ are two functions of x, then

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{a(x)}^{b(x)} f(t)\mathrm{d}t = f(b(x))b'(x) - f(a(x))a'(x)$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.