Math 1210: Calculus I
The second Fundamental Theorem of Calculus

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.4

Theorem (First FTC)

Let f be continuous on $[a, b]$, and let x be a(ny) point in (a, b). Then:

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\mathrm{d}t = f(x).$$

I.e., if $A(x) = \int_a^x f(t)\mathrm{d}t$, then

$$\frac{\mathrm{d}}{\mathrm{d}x} A(x) = f(x)$$

This is one of the major milestones in this course.

Perhaps its importance is vague since it's not clear why we care about accumulation functions.

Definite integrals

An exercise on slide `S28-D12(f)` motivates things better: suppose we wish to compute,

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and note that $A(a) = \int_a^a f(t)\mathrm{d}t = 0$. We need to compute $A(b)$.

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Suppose F is some antiderivative of f (we've spent some time computing these).

By the first FTC: A is also an antiderivative of F, so,

$$F(x) = A(x) + C,$$

for some constant C.

The "Second" FTC

$$F(x) = A(x) + C,$$

Since we know $A(a) = 0$, then we can compute C:

$$F(a) = A(a) + C \quad \Longrightarrow \quad C = F(a).$$

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So $F(x) = A(x) + F(a)$. Recall we are trying to compute $A(b)$. So now plugging in $x = b$ and solving for $A(b)$:

$$A(b) = F(b) - F(a).$$

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Theorem (Second FTC)

Let f be continuous on $[a, b]$, and let F by any antiderivative of f on $[a, b]$. Then,

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

The Second FTC, unpacked

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a), \quad F \text{ satisfies } F'(x) = f(x).$$

- We know how to compute antiderivatives for quite a few functions: The Second FTC allows us to compute definite integrals with relative ease. (Compared to using, say, Riemann sums.)

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- The function F can be *any* antiderivative. So one could choose any convenient value of the undetermined constant C when computing antiderivatives.
- The function F could be the *general* antiderivative, with an underdetermined constant C. (The constant will disappear when computing $F(b) - F(a)$.)

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- This theorem states that areas under curves are (intimately) related to slopes of tangent lines.
- By writing $F(b) - F(a) = \left[\int f(x)\mathrm{d}x\right]_a^b$, one way to write the Second FTC is,

$$\int_a^b f(x)\mathrm{d}x = \left[\int f(x)\mathrm{d}x\right]_a^b$$

Example (Example 4.4.4)

Compute

$$\int_{-1}^{2} (4x - 6x^2) \mathrm{d}x$$

(Ans: -12)

Example

Compute

$$\int_0^8 \left(x^{1/3} + x^{4/3}\right) \mathrm{d}x$$

(Ans: $12 + \frac{384}{7}$)

Example

Compute

$$\int_0^x 3\sin t \mathrm{d}t,$$

as a function of x.
(Ans: $3 - \cos x$)

Recap: the generalized power rule

Recall a "trick" with the chain rule that we used for antidifferentiation:

$$\int 15\left(3x^2\right)\left(1+x^3\right)^{14}\mathrm{d}x = \left(1+x^3\right)^{15} + C$$

We primarily used this trick for polynomials, and called it the "generalized power rule".

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A similar trick works for non-polynomial kinds of functions:

$$\int 3\cos(3x)\mathrm{d}x = \sin(3x) + C$$

$$\int (7x^6)\cos(x^7)\mathrm{d}x = \sin(x^7) + C$$

A tool for antidifferentiation: substitution

We can generalize this idea: if g and u are functions, then,

$$\frac{\mathrm{d}}{\mathrm{d}x} g(u(x)) = g'(u(x))\, u'(x),$$

$$\int g'(u(x))\, u'(x) \mathrm{d}x = g(u(x)) + C.$$

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Theorem ("u substitution" for indefinite integrals)

Let u and g be differentiable functions. Then,

$$\int g'(u(x))\, u'(x) \mathrm{d}x = g(u(x)) + C.$$

Substitution in a "new" way

Previously, we use the generalized power rule mostly by inspection:

$$\int x^2(1+x^3)^{40}\mathrm{d}x = \frac{1}{3\cdot 41}\int 3x^2\cdot 41\cdot\left(1+x^3\right)^{40}\mathrm{d}x = \frac{1}{123}(1+x^3)^{41}+C$$

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Formally, we can accomplish the same thing through a perhaps more transparent procedure:

$$\int x^2(1+x^3)^{40}\mathrm{d}x = \frac{1}{3}\int\left(1+x^3\right)^{40}3x^2\mathrm{d}x$$

By writing $u(x) = 1+x^3$, then $\mathrm{d}u = 3x^2\mathrm{d}x$.

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$$\int x^2(1+x^3)^{40}\mathrm{d}x = \frac{1}{3}\int \left(1+x^3\right)^{40} 3x^2\mathrm{d}x$$

By writing $u(x) = 1 + x^3$, then $\mathrm{d}u = 3x^2\mathrm{d}x$. This allows us to substitute back into the integral:

$$\frac{1}{3}\int (\underbrace{1+x^3}_{u})^{40}\underbrace{3x^2\mathrm{d}x}_{\mathrm{d}u} = \frac{1}{3}\int u^{40}\mathrm{d}u$$

NB: We have changed the variable of integration from x to u (!!)
This transformation is typically what people call "u substitution" or "substitution".

Example

Example

Compute

$$\int x^2(1+x^3)^{40}\mathrm{d}x,$$

using substitution.

Computing definite integrals

Once we can compute antiderivatives for complicated functions, we can compute areas:

$$\int_a^b g'(u(x))u'(x)\mathrm{d}x \overset{\text{FTC2}}{=} (g(u(x)))\Big|_{x=b} - (g(u(x)))\Big|_{x=a} = g(u(b)) - g(u(a))$$

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However, we can also exercise an actual *substitution* procedure, as with indefinite integrals:

$$\int_a^b g'(u(x))u'(x)\mathrm{d}x = \int_{u(a)}^{u(b)} g'(u)\mathrm{d}u = g(u(b)) - g(u(a))$$

Note that with substitution for definite integrals, we:

- Change $u'(x)\mathrm{d}x$ to $\mathrm{d}u$, as before.
- Rewrite any x's in the integrand in terms of u's, as before
- <u>And</u> change the limits to $a \to u(a)$, and $b \to u(b)$.

Example

Compute

$$\int_{-1}^{0} x^2(1+x^3)^{40}\mathrm{d}x,$$

using substitution.

Example (Example 4.4.13)

Compute

$$\int_{\pi^2/9}^{\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}\mathrm{d}x.$$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.