Math 1210: Calculus I
The Mean Value Theorem for Integrals and Symmetry

Department of Mathematics, University of Utah

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.5

The average values of numbers

We will explore how to define the "average" value of a function.

Given n numbers, $a_1, a_2, \ldots, a_n$, their average value is,

$$\frac{1}{n}\left(a_1 + a_2 + \cdots + a_n\right) = \frac{1}{n}\sum_{j=1}^{n} a_j.$$

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We can arrive at a suspiciously similar formula from Riemann sums: Suppose we wish to compute $\int_a^b f(x)\mathrm{d}x$.

We compute a Riemann sum using an equispaced partition of $[a, b]$ into n rectangles:

$$R_n = \sum_{j=1}^{n} f(x_j)\Delta x = \sum_{j=1}^{n} f(x_j)\frac{b-a}{n},$$

where we've chosen x_j in the interval $[x_{j-1}, x_j]$ to evaluate the rectangle height.

The average value of a function

Rewriting, the previous expression is,

$$R_n = (b-a)\,\frac{1}{n}\sum_{j=1}^{n} f(x_j) \quad\Longrightarrow\quad \frac{1}{b-a}R_n = \frac{1}{n}\sum_{j=1}^{n} f(x_j)$$

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Taking $n \uparrow \infty$, the right-hand side should be the average value of f:

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x = \text{Average value of } f \text{ over } [a,b].$$

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Definition (Average value of a function)

Suppose f is integrable over $[a,b]$. The **average value of f over** $[a,b]$ is,

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

A geometric interpretation

$$\text{Average value of } f \text{ over } [a,b] = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

Or:

$$(b-a) \times (\text{Average value of } f \text{ on } [a,b]) = \text{Area under } f \text{ on } [a,b]$$

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Note that $(b-a)$ is the width of the interval $[a,b]$, so if we interpret the average value of f as the height of a rectangle:

The left-hand side above is a width-$(b-a)$ rectangle with area equivalent to $\int_a^b f(x)\mathrm{d}x$.

I.e., if we replaced the area under the curve with a rectangle of equal width, then the average value of f is the height of this rectangle.

Example (Example 4.5.1)

Compute the average value of $f(x) = x \sin x^2$ on the interval $[0, \sqrt{\pi}]$.
(Ans: $\frac{1}{\sqrt{\pi}}$.)

The MVT for Integrals

Recall the Mean Value Theorem (for derivatives): if f is continuous and differentiable, then for any $a \neq b$:

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{for some } c \text{ between } a \text{ and } b.$$

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We can apply this result to the accumulation function for f: Define $F(x) = \int_a^x f(t)\mathrm{d}t$.

Then there is some c in $[a, b]$, such that,

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Note that:

- $F(a) = 0$, and $F(b) = \int_a^b f(x)\mathrm{d}x$
- $F'(c) = f(c)$

Hence: there is some c such that $f(c) = \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x$.

MVT for Integrals

Theorem (Mean Value Theorem for Integrals)

Suppose that f is continuous on $[a, b]$. Then there is some number c between a and b such that,

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t.$$

In other words: f achieves its average value somewhere inside the interval.

Equivalently: a rectangle of width $b-a$ and height $f(c)$ replicates the area under the curve of f.

As with our previous encounter with the MVT: There could be many values of c.

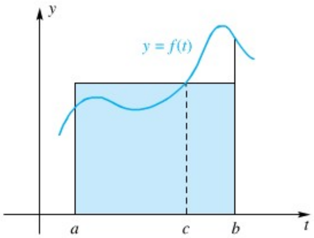


Figure 3

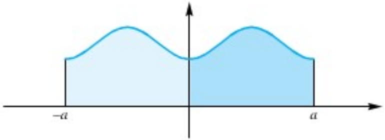
Example (Example 4.5.3)

Find all values of c that satisfy the MVT for Integrals for $f(x) = x^2$ on the interval $[-3, 3]$.
(Ans: $c = +\sqrt{3}, -\sqrt{3}$)

Symmetry for integrals

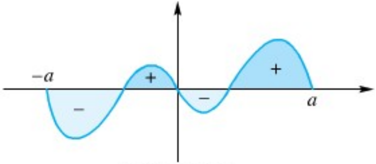
A useful tool for evaluating integrals is even/odd symmetry of functions.

For example, when integrating over $[-a, a]$:



Even function
Left area = right area

Figure 6



Odd function
Left area neutralizes right area

Figure 7

Definite integrals with symmetry

We can express the above idea through formulas:

Theorem

Suppose f is an integrable functions over $[-a, a]$. Then:

$$f \textit{ is an even function} \implies \int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x,$$

$$f \textit{ is an odd function} \implies \int_{-a}^{a} f(x)\mathrm{d}x = 0.$$

(Of course, this result is not applicable if f is neither even nor odd.)

Example (Example 4.5.5)

Evaluate $\int_{-\pi/4}^{\pi/4} \cos\left(\frac{x}{4}\right) \mathrm{d}x$.

(Ans: $4\sqrt{2}$)

Example (Example 4.5.6)

Evaluate $\int_{-5}^{5} \frac{x^5}{x^2+4} \mathrm{d}x$.
(Ans: 0)

Example (Example 4.5.7)

Evaluate $\int_{-2}^{2} \left(x \sin^4 x + x^3 - x^4\right) \mathrm{d}x$. (Ans: $-64/5$)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.