Math 1210: Calculus I
Numerical Integration

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 4.6

Evaluating integrals

We have presented the evaluation of definite integrals or areas,

$$\int_a^b f(x)\mathrm{d}x,$$

as simply an exercise in identifying an antiderivative of f and evaluating it:

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a), \qquad\qquad F'(x) = f(x).$$

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In practical situations:

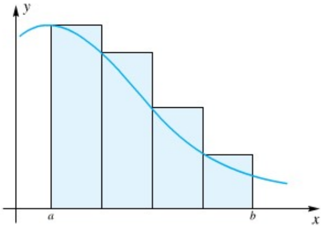
- We can't compute an antiderivative of f (e.g., $f(x) = \sin x^2$)
- We don't have a formula for f, and instead just have measurements (e.g., velocity data at regular intervals of time)

In these cases, the typical strategy is to result to approximation via numerical computation.

Riemann sums, again

The generic idea is not terribly sophisticated: we already know how to compute an approximation via a Riemann sum.

In this figure, our approximation to the area under the curve is the sum of areas of 4 rectangles.



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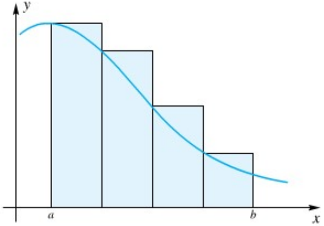


Using an equispaced partition of $[a, b]$, then this approximation uses the left-hand point as a sample point:

$$\int_a^b f(x)\mathrm{d}x \approx R_4 = \Delta x \left(f(x_0) + f(x_1) + f(x_2) + f(x_3)\right), \qquad \Delta x = \frac{b-a}{4}$$

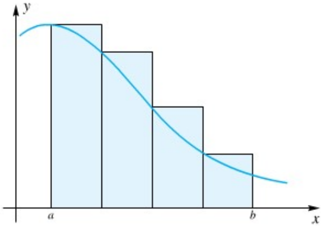
Note that this requires us to evaluate f 4 times. Presumably, if we use R_n for $n > 4$, we'll get a better approximation, but this requires us to evaluate f more times.
(Clearly there is a cost-accuracy tradeoff here.)

The Left Riemann Sum

There are a few choice of numerical integration strategy. We've already seen the first one: The **Left Riemann Sum**.

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{j=1}^{n} f(x_{j-1})\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.

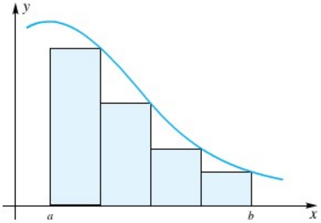


The Right Riemann Sum

We could, of course, choose the right-hand point as the sample point.
This yields the **Right Riemann Sum**.

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{j=1}^{n} f(x_j)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.

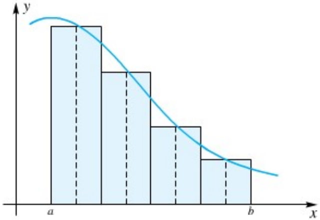


The Midpoint Riemann Sum

What about a point inside the interval? The middle seems like a reasonable choice. This yields the **Midpoint Riemann Sum**.

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.



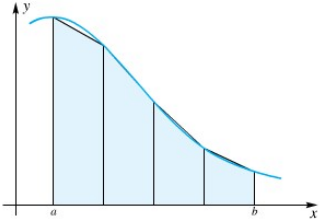
(It's hard to tell, but this overcounts on one side and undercounts on the other. Maybe it's more accurate?)

The Trapezoidal Rule

Here's a different idea: why don't we compute the area of a trapezoid using the left- and right-hand points of the interval? This is the **Trapezoidal Rule**.

$$\int_a^b f(x)\mathrm{d}x \approx \sum_{j=1}^{n} \frac{f(x_{j-1}) + f(x_j)}{2}\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.



(Visually, this rule appears to commit a smaller mistake than the others.)

Example

No matter which strategy is used, such approximations essentially require a calculator/computer.

Example (Example 4.6.1)

Approximate the integral $\int_1^3 \sqrt{4-x}\,\mathrm{d}x$ using the 4 numerical integration schemes previously introduced with $n = 4$.
(Ans: Left Riemann Sum 2.9761, Right Riemann Sum 2.6100, Midpoint Riemann Sum 2.7996, Trapezoidal Rule 2.7996. Exact answer $2\sqrt{3} - 2/3 \approx 2.797$.)

Error estimates

All these approximations commit some mistake. Without some guidance on which to use, it's hard to tell which is "better".

$$\int_a^b f(x)\mathrm{d}x = (n\text{-interval sum}) + E_n.$$

How does E_n behave for all the sums we've seen?

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How does E_n behave for all the sums we've seen?

Theorem (Numerical Integration Error Estimates)

$$E_n = \frac{(b-a)^2}{2n} f'(c) \qquad \textit{(Left Riemann sum)}$$

$$E_n = -\frac{(b-a)^2}{2n} f'(c) \qquad \textit{(right Riemann sum)}$$

$$E_n = \frac{(b-a)^3}{24n^2} f''(c) \qquad \textit{(Midpoint Riemann sum)}$$

$$E_n = -\frac{(b-a)^3}{12n^2} f'(c) \qquad \textit{(Trapezoidal Rule)}$$

where c is some number inside $[a,b]$. (It's a different number for each row above.)

NB: For all these choices, $E_n \to 0$ as $n \uparrow \infty$.
But E_n goes to zero faster for some choices.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.