Assignments for rest of semester:

- Lab on Thursday (last one)
- HW #13 due Thursday (Gradescope, 11:59pm MT)

- HW #14 due Tuesday Apr 22

Classes for rest of semester

- This week: normal
- Next week (Mon + Tues Apr 21-22): NO CLASS
- After classes end: I'll schedule 2 review sessions
  (Maybe: Wed Apr 23 + Fri Apr 25?)

Math 1210: Calculus I
Area of Plane Regions

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 5.1

Areas of regions

Now that we can compute definite integrals, we can compute areas.

$$\int_a^b f(x)\mathrm{d}x = \text{Signed area between } y = f(x) \text{ and } y = 0.$$

The operative word above is signed. In practical applications, one is often not interested in signed area, but just in area, i.e., a non-negative quantity.

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If the region we wish to compute the area of is between $x = a$ and $x = b$, above the x-axis, and bounded above by $y = f(x)$, then the area is just $\int_a^b f(x)\mathrm{d}x$.

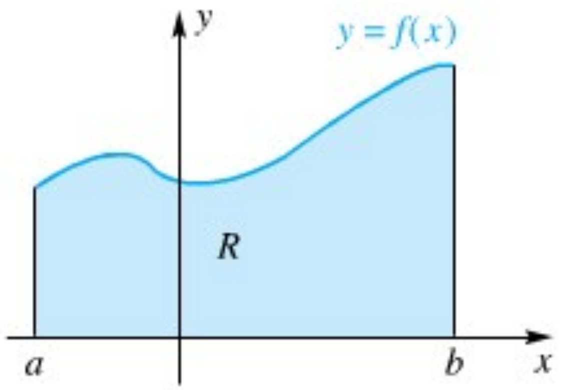


Figure 1

Area below the x axis

If the area is <u>below</u> the x-axis, most of the details are the same.

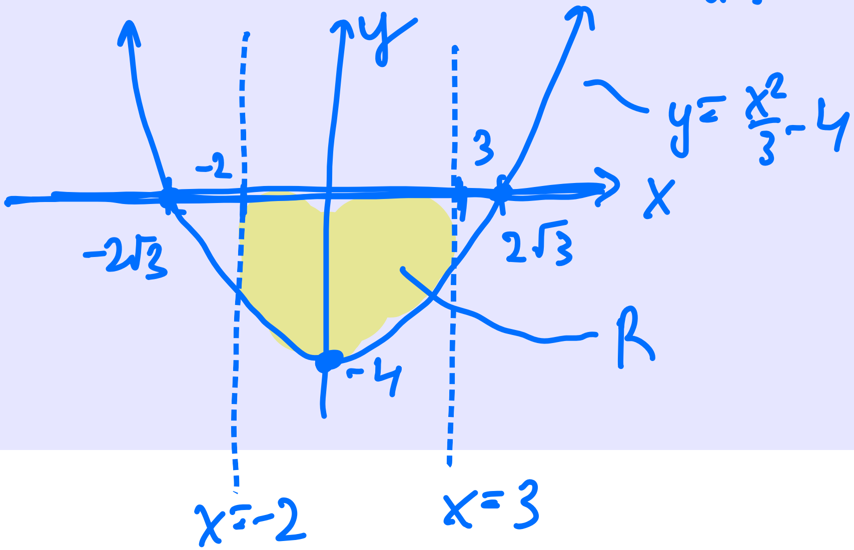
Example (Example 4.1.2)

Compute the area of the region R bounded by $y = x^2/3 - 4$, the x-axis, $x = -2$, and $x = 3$.

not signed area

not the answer → signed area: $\int_{-2}^{3}\left(\frac{x^2}{3}-4\right)dx$

(will be <0)



area: $-\int_{-2}^{3}\left(\frac{x^2}{3}-4\right)dx$

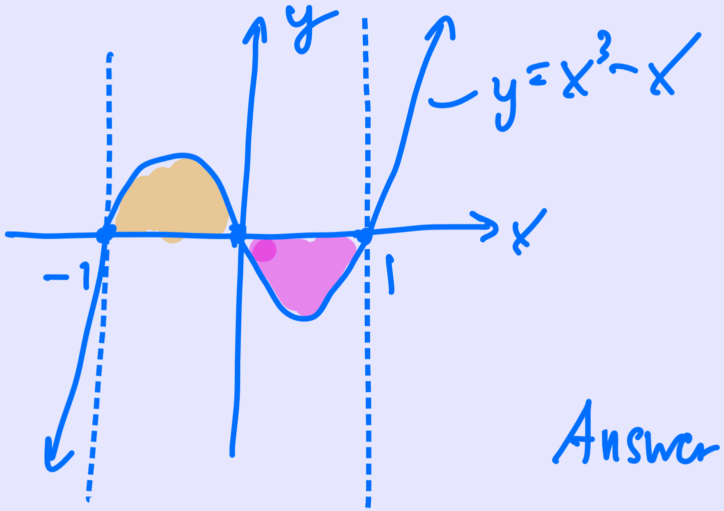
$= \ldots$

Something a bit more complicated

Example

Compute the area between $x = -1$ and $x = 1$ bounded between the x-axis and $y = x^3 - x$.

$\int_{-1}^{1} (x^3 - x)\,dx$ is <u>not</u> the answer.



: $\int_{-1}^{0} (x^3 - x)\,dx$

: $-\int_{0}^{1} (x^3 - x)\,dx$

Answer: $\int_{-1}^{0} (x^3 - x)\,dx - \underbrace{\int_{0}^{1} (x^3 - x)\,dx}_{\left(\frac{x^4}{4} - \frac{x^2}{2}\right)\Big|_0^1 = -1/4}$

Area: $\frac{1}{4} - \left(-\frac{1}{4}\right) = \frac{1}{2}$

Area between curves

D32-S05(a)

Consider graphs of two functions $y = f(x)$ and $y = g(x)$.

Suppose we want to compute the area between $x = a$ and $x = b$ that is bounded *between* the two graphs $y = f(x)$ and $y = g(x)$.

Area between curves

Consider graphs of two functions $y = f(x)$ and $y = g(x)$.

Suppose we want to compute the area between $x = a$ and $x = b$ that is bounded *between* the two graphs $y = f(x)$ and $y = g(x)$.

While actually using Riemann sums can be painful, the idea is extremely helpful!

Here, if $f(x) \geqslant g(x)$, then a picture that identifies vertical slices reveals that this sought area is,

$$A = \int_a^b (f(x) - g(x))\mathrm{d}x.$$

$$= \int_a^b f(x)dx - \int_a^b g(x)\,dx$$

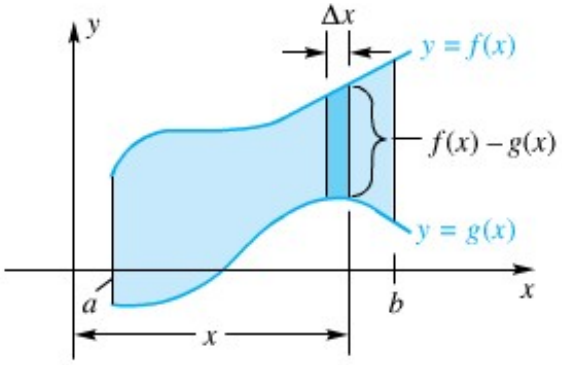
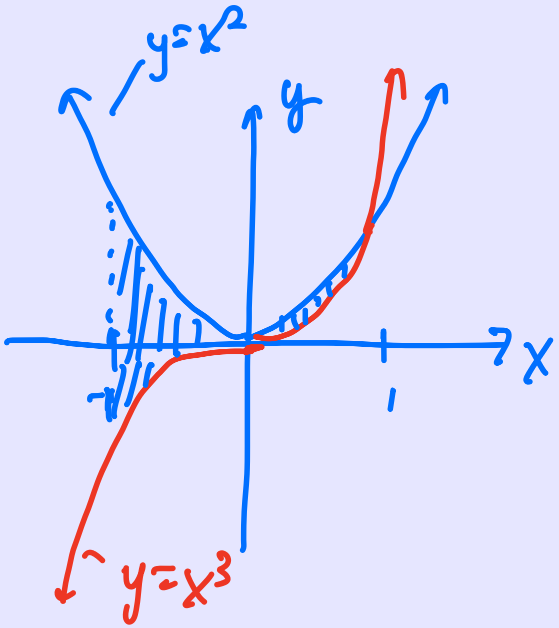


Figure 7

Example

Compute the area bounded by $x = -1$, $x = 1$, $y = x^3$ and $y = x^2$.



Area: $\int_{-1}^{0} (x^2 - x^3)\,dx$ (left piece)

$\int_{0}^{1} (x^2 - x^3)\,dx$ (right piece)

Total: $\int_{-1}^{0} (x^2 - x^3)\,dx + \int_{0}^{1} (x^2 - x^3)\,dx$

$= \int_{-1}^{1} (x^2 - x^3)\,dx$

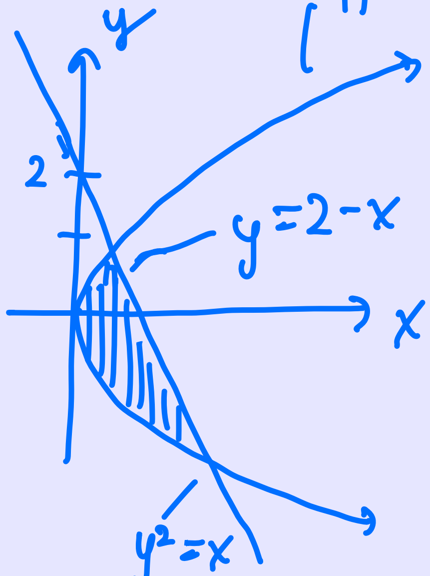
$= \ldots$ (compute number)

A trickier example

Example

Compute the area bounded by the graphs of $x = y^2$ and $y = 2 - x$.

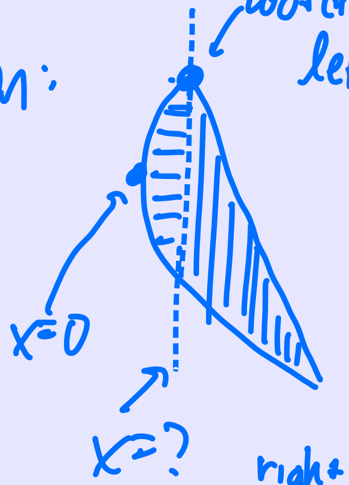
upper: $y = \sqrt{x}$



(lower: $y = -\sqrt{x}$)

One option:

coordinates = ?



left half: upper curve is $y = +\sqrt{x}$, lower curve is $y = -\sqrt{x}$

right half: upper: $y = 2 - x$, lower: $y = -\sqrt{x}$

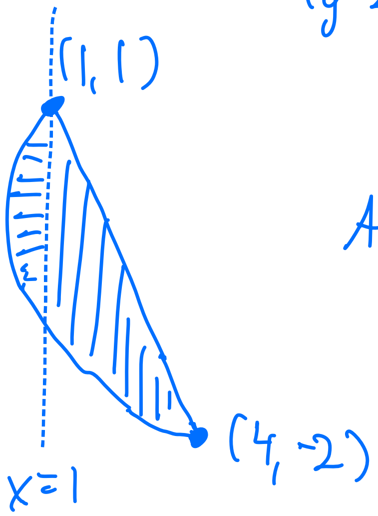
Point of intersection: $y = 2 - x, \quad x = y^2$

$$y = 2 - y^2$$

$$y^2 + y - 2 = 0$$

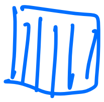
$$(y+2)(y-1) = 0 \to y = +1, \quad y = -2$$

$$x = 1 \qquad x = 4$$



Area: $\int_0^1 \left(+\sqrt{x} - -\sqrt{x}\right) dx$

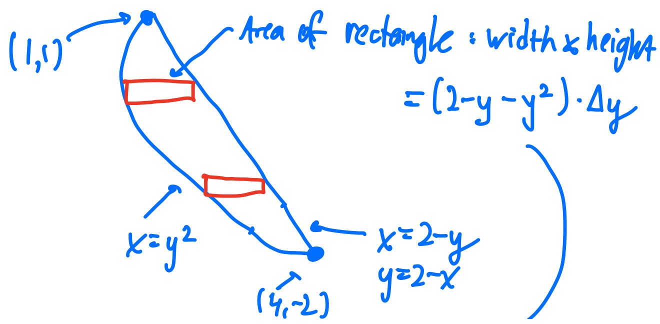
$+ \int_1^4 \left(2 - x - -\sqrt{x}\right) dx$



Area: $\int_0^1 2\sqrt{x}\, dx + \int_1^4 \left(2 - x + \sqrt{x}\right) dx$

$= \text{-- (number)}$

Another option:



limit as $\Delta y \to 0$

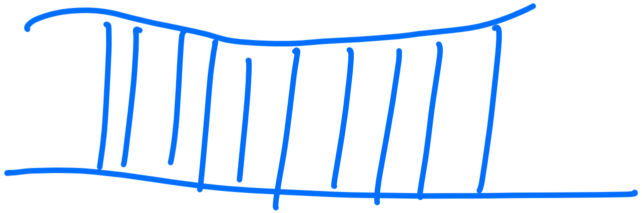
$$\int_{-2}^{1} (2-y-y^2)\,dy = \dots \text{(number)} \leftarrow \underline{\text{Area}}$$

smallest/largest y values

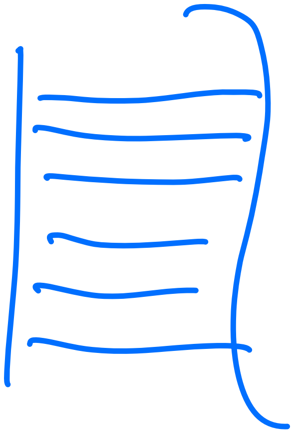
Horizontal slicing

From the previous example: graphing the region is <u>essential</u> in these problems!

When convenient, we should compute integrals corresponding to *horizontal* slices, not vertical ones!



vs



Horizontal slicing

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When computing areas of regions,

- Graph the region, and identify boundary curves/lines
- Determine if horizontal or vertical slices are more convenient or straightforward.
- Set up definite integrals corresponding to the optimal slicing strategy.

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.