Math 1210: Calculus I
Area of Plane Regions

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 5.1

Areas of regions

Now that we can compute definite integrals, we can compute areas.

$$\int_a^b f(x)\mathrm{d}x = \text{Signed area between } y = f(x) \text{ and } y = 0.$$

The operative word above is signed. In practical applications, one is often not interested in signed area, but just in area, i.e., a non-negative quantity.

Areas of regions

Now that we can compute definite integrals, we can compute areas.

$$\int_a^b f(x)\mathrm{d}x = \text{Signed area between } y = f(x) \text{ and } y = 0.$$

The operative word above is signed. In practical applications, one is often not interested in signed area, but just in area, i.e., a non-negative quantity.

If the region we wish to compute the area of is between $x = a$ and $x = b$, above the x-axis, and bounded above by $y = f(x)$, then the area is just $\int_a^b f(x)\mathrm{d}x$.

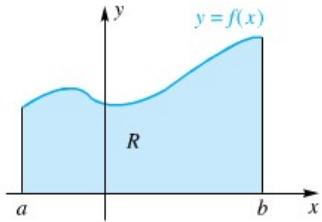

Figure 1

Area below the x axis

If the area is below the x-axis, most of the details are the same.

Example (Example 4.1.2)

Compute the area of the region R bounded by $y = x^2/3 - 4$, the x-axis, $x = -2$, and $x = 3$.

Example

Compute the area between $x = -1$ and $x = 1$ bounded between the x-axis and $y = x^3 - x$.

Area between curves

Consider graphs of two functions $y = f(x)$ and $y = g(x)$.

Suppose we want to compute the area between $x = a$ and $x = b$ that is bounded *between* the two graphs $y = f(x)$ and $y = g(x)$.

Area between curves

Consider graphs of two functions $y = f(x)$ and $y = g(x)$.

Suppose we want to compute the area between $x = a$ and $x = b$ that is bounded *between* the two graphs $y = f(x)$ and $y = g(x)$.

While actually using Riemann sums can be painful, the idea is extremely helpful!

Here, if $f(x) \geqslant g(x)$, then a picture that identifies vertical slices reveals that this sought area is,

$$A = \int_a^b (f(x) - g(x))\mathrm{d}x.$$

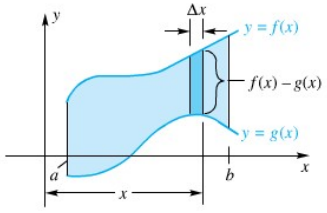

Figure 7

Example

Compute the area bounded by $x = -1$, $x = 1$, $y = x^3$ and $y = x^2$.

Example

Compute the area bounded by the graphs of $x = y^2$ and $y = 2 - x$.

Horizontal slicing

From the previous example: graphing the region is essential in these problems!

When convenient, we should compute integrals corresponding to *horizontal* slices, not vertical ones!

Horizontal slicing

From the previous example: graphing the region is essential in these problems!

When convenient, we should compute integrals corresponding to *horizontal* slices, not vertical ones!

When computing areas of regions,

- Graph the region, and identify boundary curves/lines
- Determine if horizontal or vertical slices are more convenient or straightforward.
- Set up definite integrals corresponding to the optimal slicing strategy.

References I

 Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.