Assignments for rest of semester:

- Lab on Thursday (last one)
- HW #13 due Thursday (Gradescope, 11:59pm MT)
- HW #14 due Tuesday Apr 22

Classes for rest of semester

- This week: normal
- Next Week (Mon + Tues Apr 21-22): NO CLASS
- After classes end: I'll schedule 2 review sessions
  (Maybe: Wed Apr 23 + Fri Apr 25 ?)

Wed Apr 23	Fri Apr 25
10:30-11:45	11:30-12:45
JFB 101	JFB 101

Math 1210: Calculus I
Volumes of slabs, cylinders, washers

Department of Mathematics, University of Utah

Spring 2025

Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 5.2

Volumes

We developed the definite integral to compute areas of planar regions.

One application is that we can use it to compute three-dimensional *volumes* as well.

The general idea is not too complicated: using simple geometry, we can compute volumes of "cylindrical" regions if we know the area of the base:

$$V = A \cdot h$$

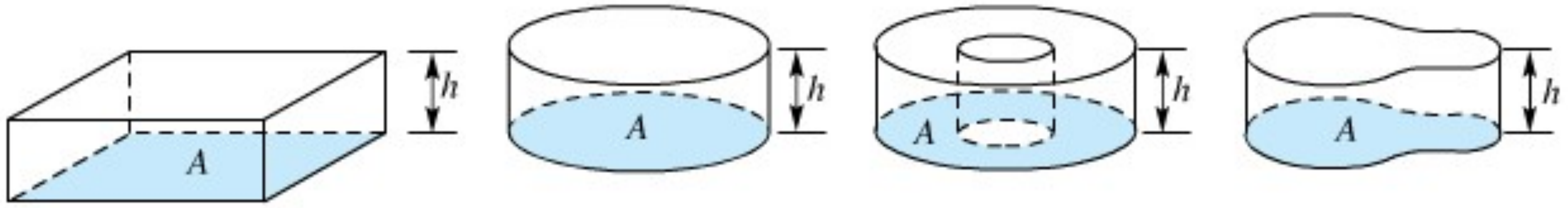

Figure 1

Cylinders of tiny height

The key tool we'll use is that we can compute volumes ΔV of cylinders of very tiny height Δh:

$$\Delta V = A\Delta h.$$

To motivate the next step, let's turn this picture on its side, so the height h is horizontal.
And then we'll rename it to x:

$$\Delta V = A\Delta x.$$

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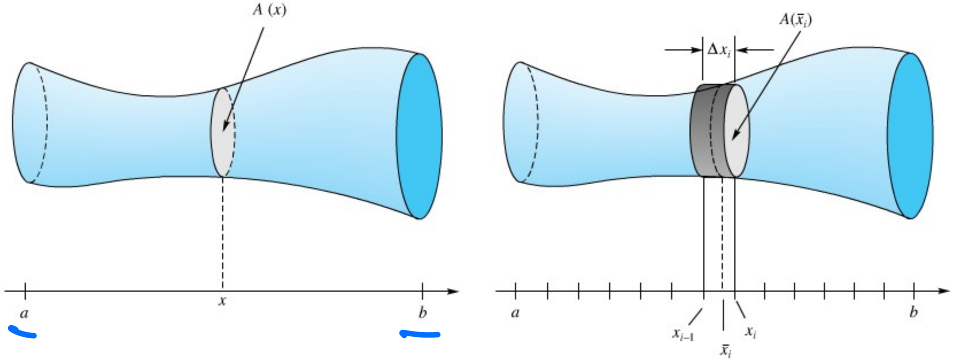
Key idea: add up small cylinder volumes to approximate the volume of a complicated region

$$\Delta V_j = A(x_j) \Delta x_j$$



Like Riemann sums, volume is the limit with infinitely many cylinders:

$$\lim_{n\to\infty} \sum_{j=1}^{n} A(x_j) \Delta x_j = \int_a^b A(x) \mathrm{d}x = V$$

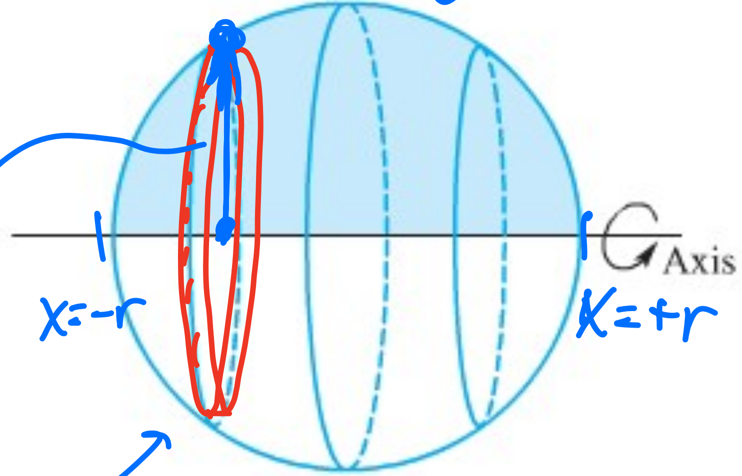
Volume of a sphere

Compute the volume of a sphere of radius r.

$x^2+y^2=r^2$

$y=\pm\sqrt{r^2-x^2}$

$y=+\sqrt{r^2-x^2}$



$$V(\cancel{x}) = \int_a^b A(x)\,dx$$

$$= \int_{-r}^{r} \pi(r^2-x^2)\,dx$$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]\Big|_{-r}^{r}$$

$$= \pi\left[r^3 - \frac{r^3}{3} - \left(-r^3 - \frac{(-r)^3}{3}\right)\right]$$

$$= \pi\left[2r^3 - \frac{2r^3}{3}\right] = 4\pi r^3/3$$

Cross-sectional Area is area of a circle

$= \pi y^2 = \pi(r^2-x^2)$

x ranges from $-r$ to $+r$

Volume of a cone

Compute the volume of a circular cone with base of radius r and height h.

$$V = \int_a^b A(y)\,dy$$

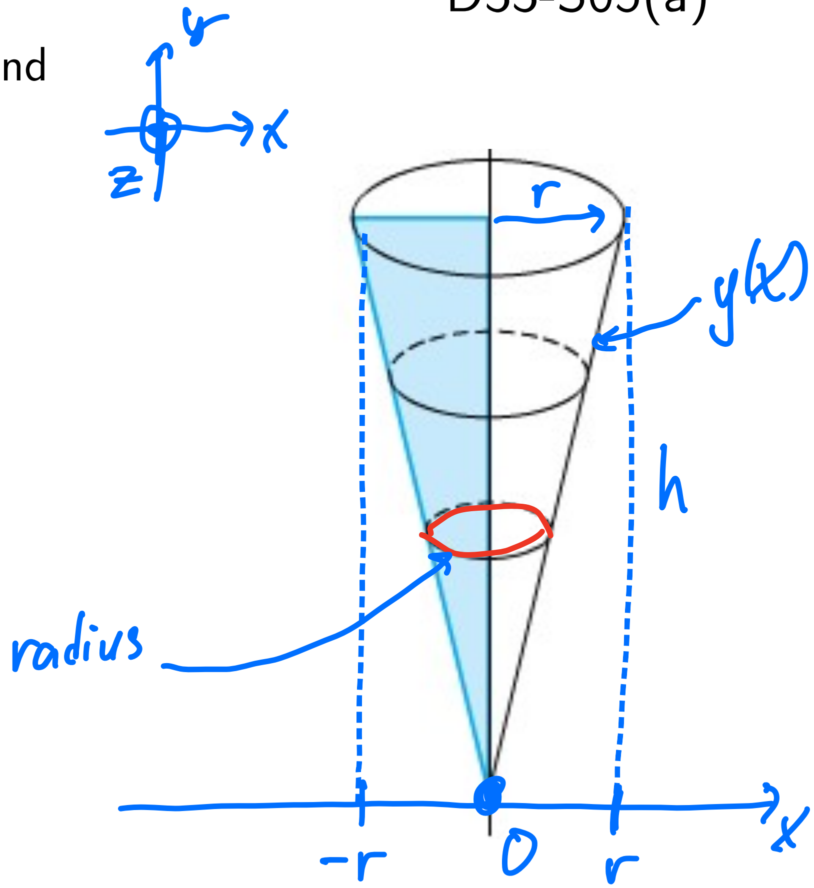
$$= \int_0^h A(y)\,dy$$

$$A(y) = \pi\,(\text{radius})^2$$

$$= \pi x^2$$

$y(x)$: line through $(0,0)$ and (r,h)

$$y = \frac{h}{r}x \rightarrow x = \frac{r}{h}y$$



$$A(y) = \pi x^2 = \pi \left(\frac{r}{h} y\right)^2 = \frac{\pi r^2}{h^2} y^2$$

$$V = \int_0^h A(y)\, dy = \int_0^h \frac{\pi r^2}{h^2} y^2 dy = \frac{\pi r^2}{h^2} \frac{y^3}{3}\Big|_0^h$$

$$= \frac{\pi r^2}{h^2} \frac{h^3}{3} = \frac{1}{3} \pi h r^2$$

Example (Example 5.2.1)

Find the volume of the solid of revolution obtained by revolving the plane region R bounded by $y = \sqrt{x}$, the x axis, and the line $x = 4$.

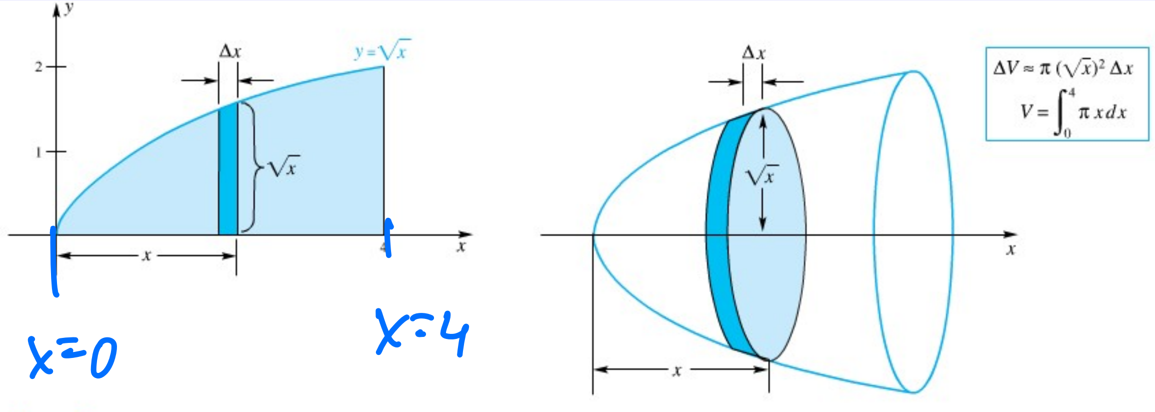


Figure 7

$V = \int_a^b A(x)\,dx = \int_0^4 A(x)\,dx$

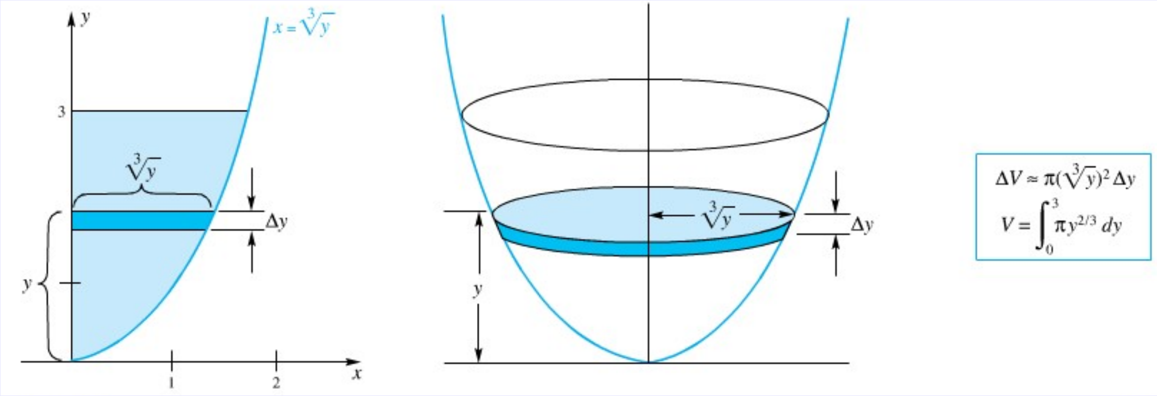
$= \int_0^4 \pi x\,dx$

$A(x)$: circle, radius $y = \sqrt{x}$

$A(x) = \pi(\sqrt{x})^2 = \pi x$

Example (Example 5.2.2)

Find the volume of the solid generated revolution generated by revolving the region bounded by the curve $y = x^3$, the y-axis, and the line $y = 3$, about the y-axis.



Method of washers

When the volume can be sliced into circular washers, a similar approach can be used.

"In some cases, cross-sectional areas are shaped like washers)

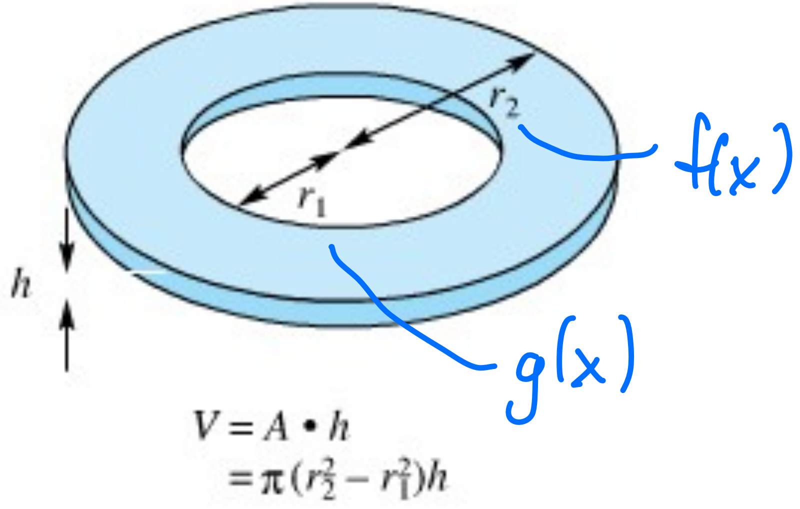


Figure 9

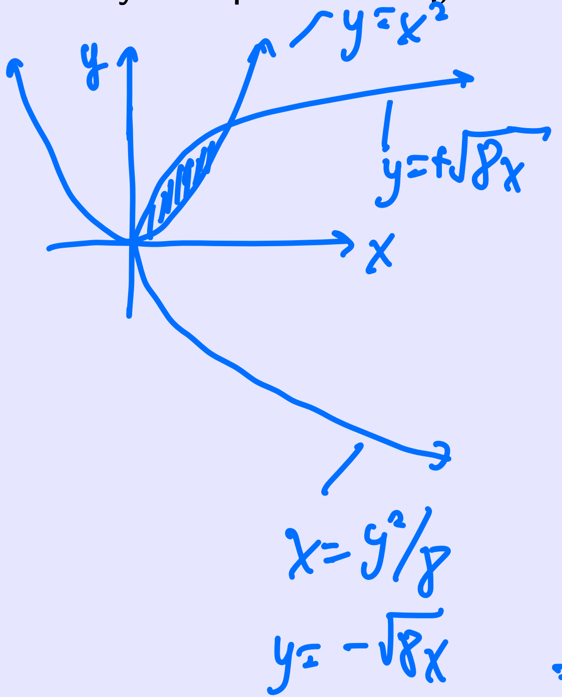
If the outer radius of the washer is $f(x)$, and the inner radius is $g(x)$, then the volume is now

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\pi(f(x_i)^2 - g(x_i)^2)\right] \Delta x_i = \int_a^b \left[\pi(f(x)^2 - g(x)^2)\right] \mathrm{d}x$$

Washers example

Example (Example 5.2.4)

Find the volume of the solid generated by revolving the region bounded by the parabolas $y = x^2$ and $y^2 = 8x$ about the x-axis.



Volume of a single washer: area × $\underbrace{\text{height}}_{\Delta x}$

$\pi(\sqrt{8x})^2 - \pi(x^2)^2$

Volume: $\pi(8x - x^4)\,\Delta x$

$\Rightarrow \int_0^{?} \pi(8x - x^4)\,dx$

?: x-coordinate of intersection of graphs

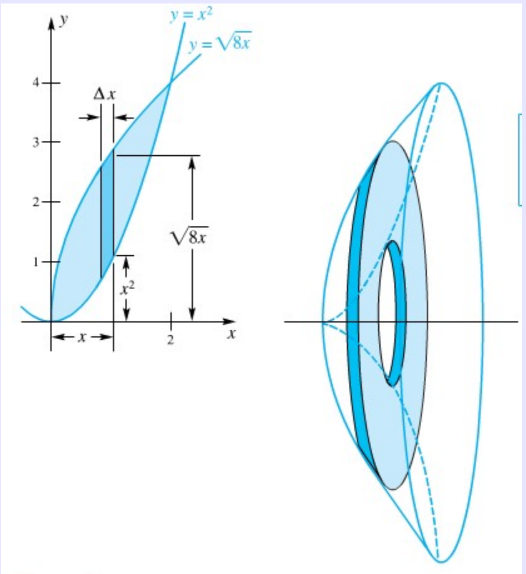


Figure 10

When does $y=x^2$ intersect $y^2=8x$?

$(0,0)$: one intersection point.

$y=x^2$, $y=\sqrt{8x}$

$x^2=\sqrt{8x} \rightarrow x^4=8x$

$x=2$ = ?

Volume: $\int_0^2 \pi(8x-x^4)\,dx$

$=$ ... (number)

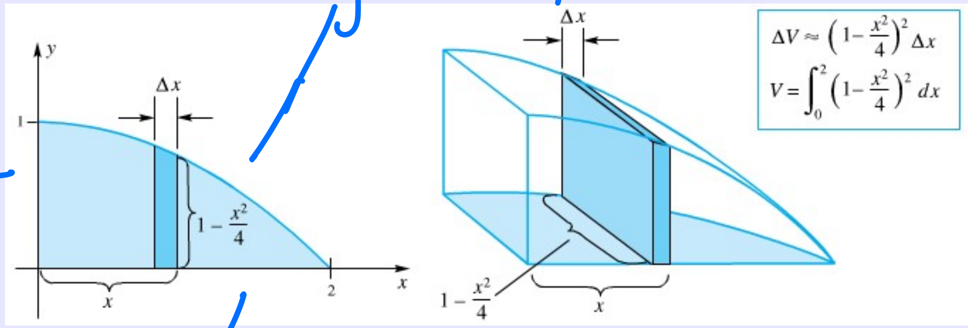
Volumes via other cross-sections

Example (Example 5.2.5)

Let the base of a solid be the first quadrant plane region bounded by $y = 1 - x^2/4$, the x-axis, and the y-axis. Suppose that cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

$y = 1 - x^2/4$

y-axis



x-axis

Volume of one slice: $\Delta x \cdot (\text{area}) = \Delta x \left(1 - x^2/4\right)^2$

Volume $\int_0^2 \left(1 - x^2/4\right)^2 dx = \cdots$ (number)

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.