Math 1210: Calculus I
Volumes of slabs, cylinders, washers

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 5.2

Volumes

We developed the definite integral to compute areas of planar regions.

One application is that we can use it to compute three-dimensional *volumes* as well.

The general idea is not too complicated: using simple geometry, we can compute volumes of "cylindrical"regions if we know the area of the base:

$$V = A \cdot h$$

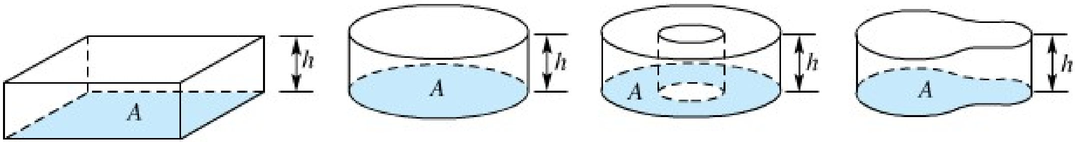

Figure 1

Cylinders of tiny height

The key tool we'll use is that we can compute volumes ΔV of cylinders of very tiny height Δh:

$$\Delta V = A\Delta h.$$

To motivate the next step, let's turn this picture on its side, so the height h is horizontal.
And then we'll rename it to x:

$$\Delta V = A\Delta x.$$

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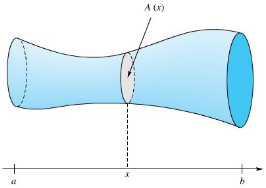
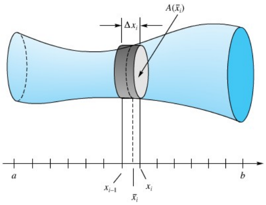
Key idea: add up small cylinder volumes to approximate the volume of a complicated region

$$\Delta V_j = A(x_j) \Delta x_j$$

Like Riemann sums, volume is the limit with infinitely many cylinders:

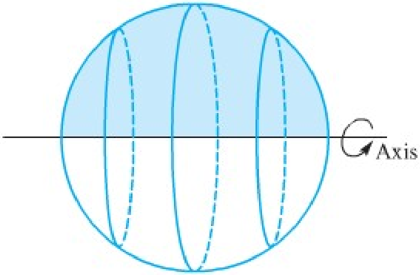




$$\lim_{n \to \infty} \sum_{j=1}^{n} A(x_j) \Delta x_j = \int A(x) \mathrm{d}x = V$$

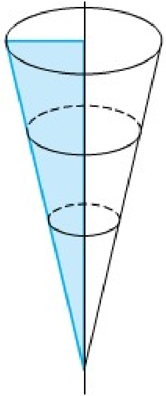
Volume of a sphere

Compute the volume of a sphere of radius r.



Volume of a cone

Compute the volume of a circular cone with base of radius r and height h.



Example (Example 5.2.1)

Find the volume of the solid of revolution obtained by revolving the plane region R bounded by $y = \sqrt{x}$, the x axis, and the line $x = 4$.

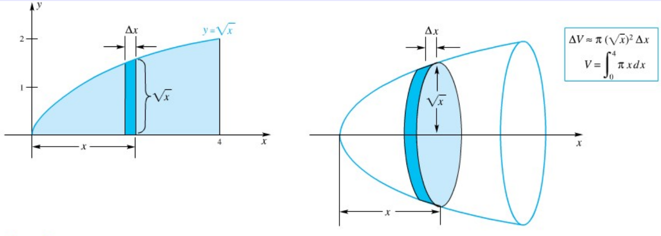
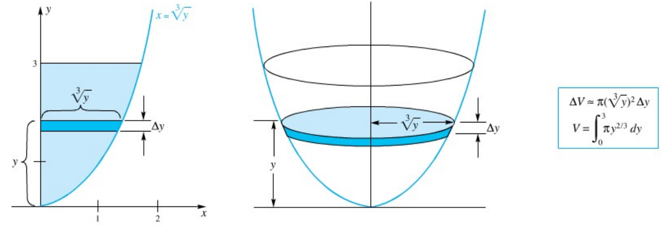


Figure 7

Example (Example 5.2.2)

Find the volume of the solid generated revolution generated by revolving the region bounded by the curve $y = x^3$, the y-axis, and the line $y = 3$, about the y-axis.



Method of washers

When the volume can be sliced into circular washers, a similar approach can be used.

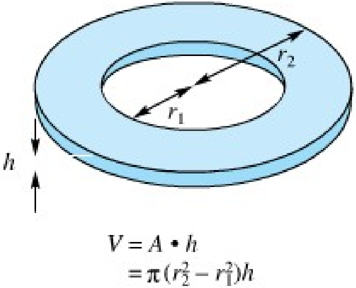


Figure 9

If the outer radius of the washer is $f(x)$, and the inner radius is $g(x)$, then the volume is now

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\pi(f(x_i)^2 - g(x_i)^2)\right] \Delta x_i = \int_a^b \left[\pi(f(x)^2 - g(x)^2)\right] \mathrm{d}x$$

Washers example

Example (Example 5.2.4)

Find the volume of the solid generated by revolving the region bounded by the parabolas $y = x^2$ and $y^2 = 8x$ about the x-axis.

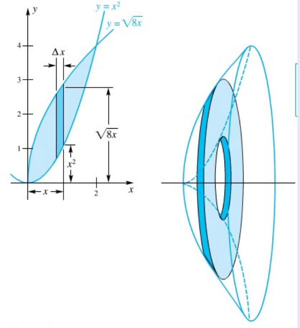
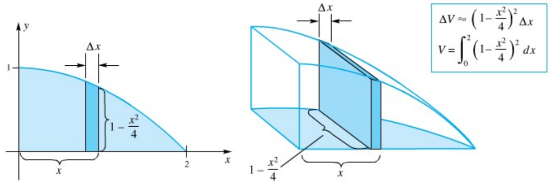


Figure 10

Example (Example 5.2.5)

Let the base of a solid be the first quadrant plane region bounded by $y = 1 - x^2/4$, the x-axis, and the y-axis. Suppose that cross sections perpendicular to the x-axis are squares. Find the volume of the solid.



References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.