Math 1210: Calculus I
Volumes of solids of revolution: shells

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Accompanying text: Varberg, Purcell, and Rigdon 2007, Section 5.3

Volumes: shells

In the previous section: we used integrals to compute volumes of revolution by summing up volumes of small discs or washers.

There is another useful, complementary strategy: summing up volumes of small cylindrical shells.

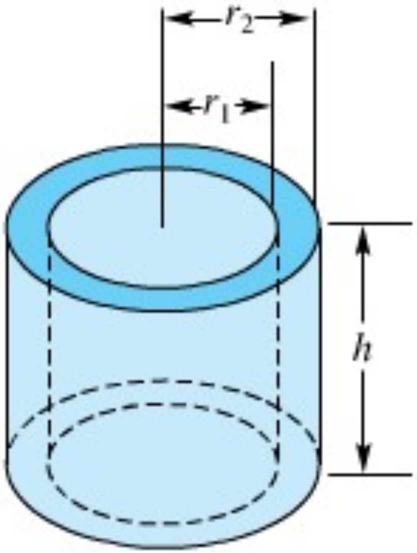

Figure 1

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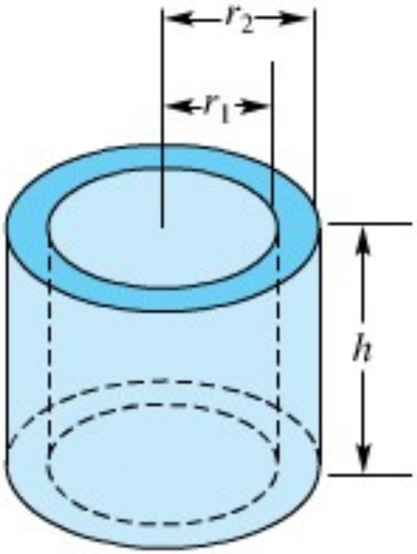
The volume of this shell is

$$V = \underbrace{\left(\pi r_2^2 - \pi r_1^2\right)}_{\text{cross-sectional area}} h = 2\pi h \underbrace{\left(r_2 - r_1\right)}_{\Delta r} \underbrace{\left(\frac{r_1 + r_2}{2}\right)}_{r_1 \to r_2 = r}$$

$$\frac{r + r}{2} = r$$

$$V \simeq 2\pi h r \Delta r$$



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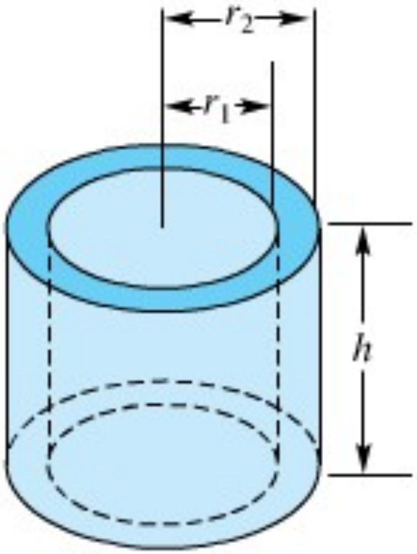


Figure 1

To create a small shell: set $r_2 - r_1 = \Delta r$.

Then set $r = \frac{1}{2}(r_1 + r_2)$, the average of the radii.

$$V = 2\pi r h \Delta r$$

One way to remember this: $2\pi r$ is the circumference of the circle. h is the height. Δr is the thickness.

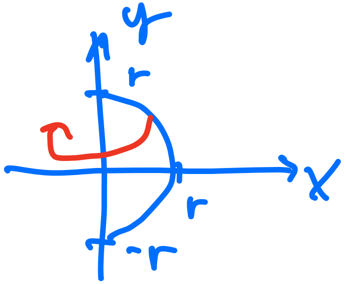
Volume of a sphere, redux

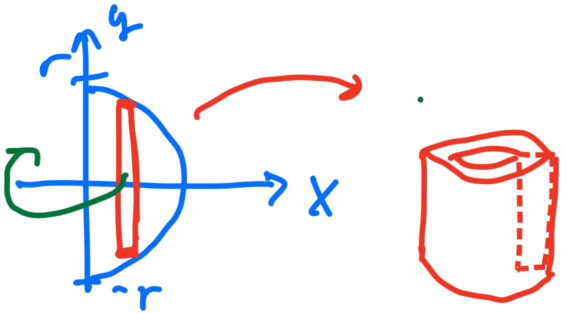
Once we have the volume of one shell, we can compute volumes by adding up these volumes.

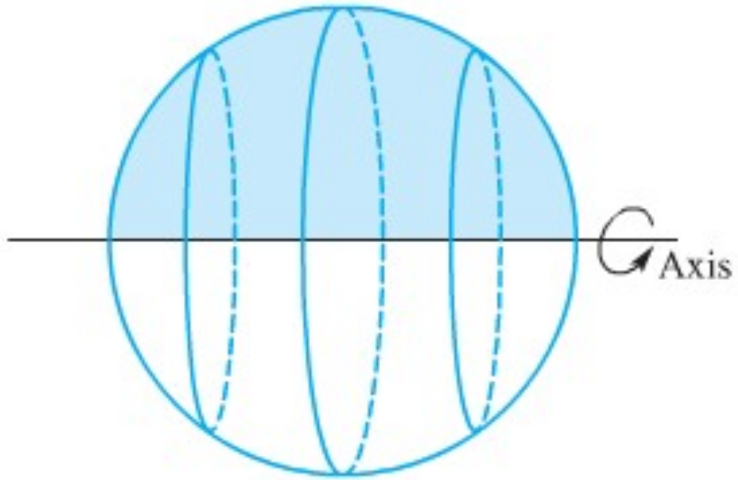
This is the "method of shells".

Compute the volume of a sphere of radius r using the method of shells.

Sphere: revolving a semicircle around y-axis



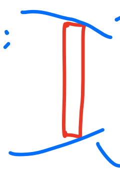




Volume of shell: $2\pi r h \Delta r$

radius r: x

width: Δr: Δx

height: 

$y=+\sqrt{r^2-x^2}$

$y=-\sqrt{r^2-x^2}$

$h=2\sqrt{r^2-x^2}$

Volume of shell: $2\pi r h \Delta r = 4\pi x\sqrt{r^2-x^2}\,dx$

Volume: $\int_0^r 4\pi x\sqrt{r^2-x^2}\,dx$

$$=4\pi\int_0^r x\sqrt{r^2-x^2}\,dx$$

$$u=r^2-x^2$$

$$du=-2x\,dx$$

$$=4\pi\int_{r^2}^{0} x\sqrt{u}\,\frac{du}{-2x}$$

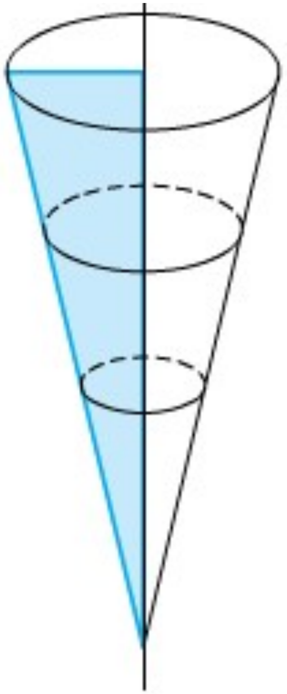
$$=2\pi\int_0^{r^2}\sqrt{u}\,du$$

$$=2\pi\,\frac{2}{3}u^{3/2}\Big|_0^{r^2}$$

$$=\frac{4\pi}{3}r^3$$

Volume of a cone, redux

Compute the volume of a circular cone with base of radius r and height h using the method of shells.



Shells example

Example (Example 5.3.3)

Find the volume of the solid generated by revolving the region in the first quadrant that is above the parabola $y = x^2$ and below the parabola $y = 2 - x^2$ about the y-axis.

Shells example 2

Example

Consider the planar region bounded by $x = 0$, $y = 0$, and $y = 1 + 2x - x^2$. Set up and evaluate an integral for the volume of the solid that results when the region is revolved around,

(a) the x-axis

(b) the y-axis

(c) the line $y = -1$

(d) the line $x = 4$

References I

Varberg, D.E., E.J. Purcell, and S.E. Rigdon (2007). *Calculus*. 9th. Pearson Prentice Hall. ISBN: 978-0-13-142924-6.