Math 6610: Analysis of Numerical Methods, I
Linear algebraic preliminaries

Department of Mathematics, University of Utah

Fall 2026

Resources:	Trefethen and Bau 1997, Lectures 1, 2, 3
	Atkinson 1989, Sections 7.1, 7.3
	Salgado and Wise 2022, Sections 1.1, 1.2

Notation

We'll use some standard math notation

- $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$
- $\in$, $\forall$, $\exists$, !
- $\{x \in \mathbb{C} \mid \mathsf{Im}\{x\} \in \mathbb{N}\}$
- $z = x + iy$ for $x, y \in \mathbb{R} \Longrightarrow \bar{z} = z^* \coloneqq x - iy$

Vectors, matrices, etc:

- $\mathbf{u} \in \mathbb{C}^n$
- $\mathbf{A} \in \mathbb{C}^{m \times n}$
- linear independence
- rank
- (conjugate) transpose
- determinant
- matrix inverse
- subspaces defined by $\mathbf{A}$: range, kernel, cokernel, corange

Hilbertian structure

$\mathbb{C}^n$ endowed with the standard inner product is a Hilbert space. If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$,

- $\langle \mathbf{u}, \mathbf{v} \rangle$, $\|\mathbf{u}\|$
- $\angle(\mathbf{u}, \mathbf{v})$
- $\mathbf{u} \perp \mathbf{v}$
- $\mathsf{Proj}_{\mathbf{v}} \mathbf{u}$
- orthogonal and orthonormal sets

All the above is also well-defined in $\mathbb{R}^n$.

Matrices and matrix algebra

An $m \times n$ matrix $\mathbf{A}$ is a tableau of elements (from $\mathbb{R}$ or $\mathbb{C}$):

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m \times n}, \qquad (A)_{j,k} = a_{j,k}, \qquad j \in [m], k \in [n]$$

A matrix is "just" a vector with "2D" indices.

Matrices come with a natural algebra, i.e., sum and product operations involving matrices:

- Product of a scalar and a matrix
- Sum of two matrices (of the same size)
- Product of two matrices (of conforming sizes)

$$\mathbf{A} \in \mathbb{C}^{m \times n}, \mathbf{B} \in \mathbb{C}^{n \times k} \quad \Longrightarrow \quad \mathbf{AB} \in \mathbb{C}^{m \times k}, \;\; (AB)_{j,k} = \sum_{\ell=1}^{n} a_{j\ell} b_{\ell k}.$$

The "four fundamental subspaces"

Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ be given. The four fundamental subspaces are uniquely defined:

- $\mathbb{C}^m \supset \mathscr{R}(\mathbf{A}) = \mathsf{range}(\mathbf{A}) = \mathsf{Im}(\mathbf{A})$, the "column space" of $\mathbf{A}$
- $\mathbb{C}^n \supset \mathscr{R}(\mathbf{A}^*) = \mathsf{corange}(\mathbf{A})$, the "row space" or "corange" of $\mathbf{A}$.
- $\mathbb{C}^n \supset \mathscr{K}(\mathbf{A}) = \mathsf{ker}(\mathbf{A})$, the "nullspace" or "kernel" of $\mathbf{A}$
- $\mathbb{C}^m \supset \mathscr{K}(\mathbf{A}^*) = \mathsf{coker}(\mathbf{A})$, the "left nullspace" or "cokernel" of $\mathbf{A}$.

Essentially by definition: $\mathscr{K}(\mathbf{A})$ contains all vectors $\mathbf{v}$ satisfying $\mathbf{A}\mathbf{v} = \mathbf{0}$. I.e., if $\mathbf{w}_1, \ldots, \mathbf{w}_m \in \mathbb{C}^n$ are conjugate-transposed rows of $\mathbf{A}$, then $\mathbf{v}$ is orthogonal to $\mathsf{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$.

Theorem (Fundamental Theorem of Linear Algebra)

For any $\mathbf{A} \in \mathbb{C}^{m \times n}$,

$$\begin{aligned} n &= \dim \mathsf{corange}(\mathbf{A}) + \dim \mathsf{ker}(\mathbf{A}), & \mathsf{corange}(\mathbf{A}) &\perp \mathsf{ker}(\mathbf{A}), & \mathbb{C}^n &= \mathsf{corange}(\mathbf{A}) \oplus \mathsf{ker}(\mathbf{A}) \\ m &= \dim \mathsf{range}(\mathbf{A}) + \dim \mathsf{coker}(\mathbf{A}), & \mathsf{range}(\mathbf{A}) &\perp \mathsf{coker}(\mathbf{A}), & \mathbb{C}^m &= \mathsf{range}(\mathbf{A}) \oplus \mathsf{coker}(\mathbf{A}) \end{aligned}$$

Norms

Metrizing linear spaces is a big business in mathematics.

Given a vector space V, a map $\|\cdot\| : V \to \mathbb{R}$ is a *norm* if it satisfies all the following properties:

- $\|\mathbf{x}\| \geq 0 \;\; \forall \mathbf{x} \in V$
- $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \;\; \forall \mathbf{x}, \mathbf{y} \in V$
- $\|c\mathbf{x}\| = |c|\|\mathbf{x}\| \;\; \forall \mathbf{x} \in V,\; c \in \mathbb{C}$.

We are mostly concerned with standard examples $V = \mathbb{R}^n, \mathbb{C}^n, \mathbb{C}^{m \times n}$, etc.

Vector norms

The "standard" examples of vector norms are the ℓ^p norms.

With $\mathbf{x} \in \mathbb{C}^n$:

$$\begin{aligned} \|\mathbf{x}\|_p^p &:= \sum_{j \in [n]} |x_j|^p, & p &\in [1, \infty) \\ \|\mathbf{x}\|_\infty &:= \max_{j \in [n]} |x_j|, & p &= \infty. \end{aligned}$$

Example

Show that $\|\cdot\|_2$ on $\mathbb{C}^n$ is a norm.

Matrix norms

One straightforward class of matrix norms consists of "entrywise" norms:

$$\|\mathbf{A}\|_{p,p} := \|\mathsf{vec}(\mathbf{A})\|_p, \qquad\qquad p \in [1, \infty],$$

where $\mathsf{vec}(\cdot)$ is the *vectorization* function.

There are "mixed" entrywise norm definitions, corresponding to taking ℓ^p vector norms of each column, and then a vector ℓ^q norm of the resulting vector of norms,

$$\|\mathbf{A}\|_{p,q} := \left(\sum_{j \in [n]} \left(\sum_{i \in [m]} |a_{i,j}|^p \right)^{q/p} \right)^{1/q}.$$

A particularly useful entrywise norm is the *Frobenius norm*,

$$\|\mathbf{A}\|_F := \|\mathbf{A}\|_{2,2}.$$

Induced matrix norms

A more conceptual collection of matrix norms are *induced* by vector norms.

By viewing $\mathbf{A} \in \mathbb{C}^{m \times n}$ as the mapping $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$, norms can be defined as the maximum relative "size" of this mapping:

$$\|\mathbf{A}\|_p \coloneqq \sup_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \qquad p \in [1, \infty].$$

(Note that $\mathbf{A}$ can be rectangular here.)

That these are proper norms is direct from the fact that $\|\cdot\|_p$ is a norm on $\mathbb{C}^m$.

Norm equivalence

A rather important and useful fact is that any two norms on the same finite-dimensional vector space are *equivalent*.

Theorem (All norms on a finite-dimensional space are equivalent)

Let V be an n-dimensional vector space, and let $\|\cdot\|_*$ and $\|\cdot\|_+$ be any two norms on this space. Then there are strictly positive constants c and k such that for all $\mathbf{x} \in V$,

$$c\|\mathbf{x}\|_* \leq \|\mathbf{x}\|_+ \leq k\|\mathbf{x}\|_*.$$

The constants c and k can depend on V (in particular n) and the choice of $\|\cdot\|_*$ and $\|\cdot\|_+$, but not on $\mathbf{x}$.

Note that the above applies equally to spaces containing vectors or matrices.

The good news: Norm equivalence suggests it doesn't matter which norm you pick.

The bad news: To prove something, it typically matters which norm you pick.

A zoo of norms, formulas, and equivalences: Vector norm bounds

Example

Compute c and k such that,

$$c\|\mathbf{x}\|_1 \leq \|\mathbf{x}\|_2 \leq k\|\mathbf{x}\|_1, \qquad \forall\, \mathbf{x} \in \mathbb{C}^n.$$

Also, identify examples of vectors $\mathbf{x}$ that achieve the upper and lower bounds above.

A zoo of norms, formulas, and equivalences: Matrix norm bounds

Example

Compute c and k such that,

$$c\|\mathbf{A}\|_1 \leq \|\mathbf{A}\|_2 \leq k\|\mathbf{A}\|_1, \qquad \forall\ \mathbf{A} \in \mathbb{C}^{m\times n}$$

Also, identify examples of matrices $\mathbf{A}$ that achieve the upper and lower bounds above.

A zoo of norms, formulas, and equivalences: A matrix 2-norm

Example

Compute $\|\mathbf{A}\|_2$, where,

$$\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 1 & 2 \end{pmatrix}$$

Hilbertian structure

Of special interest are the norms arising from inner products: these norms induce Euclidean-like geometry (Hilbert spaces).

The prototypical example on $\mathbb{C}^n$ is the ℓ^2 norm: for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, we have,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^* \mathbf{x} = \sum_{i \in [n]} x_i y_i^*, \qquad\qquad \|\mathbf{x}\|_2^2 = \langle \mathbf{x}, \mathbf{x} \rangle .$$

Inner products are *bilinear* forms over $\mathbb{R}$ and *sesquilinear* forms over $\mathbb{C}$.

One of the most useful algebraic properties of inner products that give rise to a norm $\| \cdot \|$ is the *Cauchy-Schwarz* inequality:

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

From this property one can observe that the following geometric structure of elements $\mathbf{x}, \mathbf{y}$ is reasonable:

$$\cos\left(\angle(\mathbf{x}, \mathbf{y})\right) \coloneqq \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \, \|\mathbf{y}\|}, \qquad\qquad \mathbf{x} \perp \mathbf{y} \text{ iff } \langle \mathbf{x}, \mathbf{y} \rangle = 0.$$

There are Hilbertian norms even for matrices, with a common example being the Frobenius norm:

$$\langle \mathbf{A}, \mathbf{B} \rangle_F \coloneqq \mathsf{Tr}\left(\mathbf{B}^* \mathbf{A}\right), \qquad\qquad \|\mathbf{A}\|_F^2 = \langle \mathbf{A}, \mathbf{A} \rangle_F$$

The Pythagorean Theorem

The sledgehammer killing an ant way to prove the Pythagorean Theorem: Let $\mathbf{x}_1$, $\mathbf{x}_2$ be two orthogonal vectors (say in $\mathbb{C}^n$).

Since $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0$, then

$$\begin{aligned}
\|\mathbf{x}_1 + \mathbf{x}_2\|_2^2 &= \langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2 \rangle \\
&= \langle \mathbf{x}_1, \mathbf{x}_1 \rangle + \langle \mathbf{x}_2, \mathbf{x}_2 \rangle + \underbrace{\langle \mathbf{x}_1, \mathbf{x}_2 \rangle}_{0} + \underbrace{\langle \mathbf{x}_2, \mathbf{x}_1 \rangle}_{0} \\
&= \|\mathbf{x}_1\|_2^2 + \|\mathbf{x}_2\|_2^2 .
\end{aligned}$$

One of the more useful extensions of this (not apparent from $n = 2$) is: If $\mathbf{x}_1, \ldots \mathbf{x}_k$ are k mutually orthogonal vectors in $\mathbb{C}^n$, then,

$$\left\| \sum_{j \in [k]} \mathbf{x}_j \right\|_2^2 = \sum_{j \in [k]} \|\mathbf{x}_j\|_2^2$$

References

Atkinson, Kendall (1989). *An Introduction to Numerical Analysis*. New York: Wiley. ISBN: 978-0-471-62489-9.

Salgado, Abner J. and Steven M. Wise (2022). *Classical Numerical Analysis: A Comprehensive Course*. Cambridge: Cambridge University Press. ISBN: 978-1-108-83770-5. DOI: 10.1017/9781108942607.

Trefethen, Lloyd N. and David Bau (1997). *Numerical Linear Algebra*. SIAM: Society for Industrial and Applied Mathematics. ISBN: 0-89871-361-7.