Math 6610: Analysis of Numerical Methods, I
Orthogonal, Projection, and Permutation matrices

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Resources: Trefethen and Bau 1997, Lectures 2, 6
Atkinson 1989, Sections 7.1
Salgado and Wise 2022, Sections 1.1, 1.2, 5.2

Unitary and orthogonal matrices

A *very* special and important class of matrices:

Definition (Unitary/orthogonal matrices)

A matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ is **unitary** if $\mathbf{U}^* \mathbf{U} = \mathbf{I}_n$

A matrix $\mathbf{U} \in \mathbb{R}^{n \times n}$ is **orthogonal** if $\mathbf{U}^T \mathbf{U} = \mathbf{I}_n$

A unitary matrix is one whose columns are an orthonormal basis for $\mathbb{C}^n$ under the $\ell^2(\mathbb{C}^n)$ inner product.

Theorem

Let $\mathbf{U} \in \mathbb{C}^{n \times n}$ be a unitary matrix. Then

- $\mathbf{U}^{-1} = \mathbf{U}^*$
- $\mathbf{U}\mathbf{U}^* = \mathbf{I}$
- $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$.

This last property in particular shows: unitary matrices are *isometries*, i.e., are norm-preserving.
Unitary matrices have nice geometric interpretations: they are precisely rigid rotations and/or reflections.

Projections and projection matrices: Geometric construction

With $\mathbb{C}^n$ the ambient space, we want to define square matrix "projections". Informally, we want projections to

- Act like the identity on some subspace (the range)
- Annihilate components in another subspace (the kernel)

In order to be well-defined, we need an additional condition.

Definition (Geometric definition of projection matrices)

A matrix $\mathbf{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if

1. $\mathbf{P}\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V = \mathsf{range}(\mathbf{P})$
2. $\mathbf{P}\mathbf{w} = \mathbf{0}$ for all $\mathbf{w} \in W = \mathsf{ker}(\mathbf{P})$
3. $\mathsf{range}(\mathbf{P}) \oplus \mathsf{ker}(\mathbf{P}) = \mathbb{C}^n$ and $\mathsf{range}(\mathbf{P}) \cap \mathsf{ker}\,\mathbf{P} = \{0\}$.

We typically say that $\mathbf{P}$ projects *onto* V, and projects *along* W.

This last condition is *necessary*:

- A projector $\mathbf{P}$ in $\mathbb{R}^3$ that projects onto $\mathsf{span}\{(1,0,0)\}$ along $\mathsf{span}\{(0,0,1)\}$ is undefined for input $(0,1,0)$.
- A projector $\mathbf{P}$ in $\mathbb{R}^2$ that projects onto $\mathbb{R}^2$ along $\mathsf{span}\{(1,0)\}$ is ill-defined.

Projections exist and are unique

Does this definition make sense?

Theorem

If V and W are $\mathbb{C}^N$-subspaces such that $V \cap W = \{\mathbf{0}\}$ and $\dim V + \dim W = N$, then $\exists\ !$ projection matrix $\mathbf{P} \in \mathbb{C}^{N\times N}$ such that $\mathsf{range}(\mathbf{P}) = V$ and $\ker(\mathbf{P}) = W$.

This theorem ensures that any construction for $\mathbf{P}$ identifies the unique projector.

Let $n = \dim V$.
If $\mathbf{V} \in \mathbb{C}^{N\times n}$ satisfies $\mathsf{range}(\mathbf{V}) = V$, and $\mathbf{W} \in \mathbb{C}^{N\times(N-n)}$ satisfies $\mathsf{range}(\mathbf{W}) = W$, then

$$\mathbf{P} = \left[\mathbf{V}\ \ \mathbf{0}_{N\times(N-n)}\right] \left[\mathbf{V}\ \ \mathbf{W}\right]^{-1}$$

Projection matrices: Algebraic definitions

There is a more algebraically convenient characterization of projection matrices.

A square matrix $\mathbf{A}$ is *idempotent* if $\mathbf{A} = \mathbf{A}^2$.

Theorem

$\mathbf{P} \in \mathbb{C}^{n \times n}$ is a projection matrix iff it is idempotent.

This motivates the more common definition of a projection matrix:

Definition (Algebraic definition of projection matrices)

$\mathbf{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if $\mathbf{P} = \mathbf{P}^2$.

Geometry for orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.

However, projecting along $\mathsf{range}(\mathbf{P})^\perp$ is a norm non-expansive operation, i.e.,

$$\mathsf{ker}(\mathbf{P}) = \mathsf{range}(\mathbf{P})^\perp \qquad \Longrightarrow \qquad \|\mathbf{P}\mathbf{v}\|_2 \leq \|\mathbf{v}\|_2.$$

Projection matrices $\mathbf{P}$ satisfying $\mathsf{ker}(\mathbf{P}) = (\mathsf{range}\mathbf{P})^\perp$ are *orthogonal* projections.

Definition (Geometric definition of orthogonal projection matrices)

Let $\mathbf{P}$ be a projection matrix. It's an orthogonal projector if $\mathsf{ker}(\mathbf{P}) = \mathsf{range}(\mathbf{P})^\perp$.

Orthogonal projectors

There is also an algebraically convenient characterization of orthogonal projection matrices.

Theorem

Let $\mathbf{P} \in \mathbb{C}^{n\times n}$ be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.

This motivates the more common definition of an orthogonal projection matrix:

Definition (Algebraic definition of orthogonal projectors)

$\mathbf{P} \in \mathbb{C}^{n\times n}$ is an <u>orthogonal projection matrix</u> if $\mathbf{P} = \mathbf{P}^2$ and $\mathbf{P} = \mathbf{P}^*$.

Permutation

A final class of matrices we'll consider is permutation matrices.

Definition

For a fixed $n \in \mathbb{N}$, $\pi : [n] \to [n]$ is a permutation if it is a bijection.

Every permutation of the set of indices of a vector corresponds to a linear operator on $\mathbb{C}^n$, so it can be encoded by an $n \times n$ matrix.

Definition

$\mathbf{P} \in \mathbb{C}^{n \times n}$ is a permutation matrix if there is a permutation map π on $[n]$ such that $\mathbf{P}\mathbf{e}_j = \mathbf{e}_{\pi(j)}$ for all $j \in [n]$.

Permutation matrices are orthogonal/unitary, and hence if $\mathbf{P}$ is a permutation matrix, then $\mathbf{P}^* = \mathbf{P}^{-1}$.

The space of permutation matrices is closed under matrix multiplication, so that if $\mathbf{P}$ and $\mathbf{Q}$ are permutation matrices, then so is $\mathbf{PQ}$.

References

Atkinson, Kendall (1989). *An Introduction to Numerical Analysis*. New York: Wiley. ISBN: 978-0-471-62489-9.

Salgado, Abner J. and Steven M. Wise (2022). *Classical Numerical Analysis: A Comprehensive Course*. Cambridge: Cambridge University Press. ISBN: 978-1-108-83770-5. DOI: 10.1017/9781108942607.

Trefethen, Lloyd N. and David Bau (1997). *Numerical Linear Algebra*. SIAM: Society for Industrial and Applied Mathematics. ISBN: 0-89871-361-7.