Math 6610: Analysis of Numerical Methods, I
Orthogonal, Projection, and Permutation matrices

Department of Mathematics, University of Utah

Fall 2026

Resources: Trefethen and Bau 1997, Lectures 2, 6
Atkinson 1989, Sections 7.1
Salgado and Wise 2022, Sections 1.1, 1.2, 5.2

Unitary and orthogonal matrices

A *very* special and important class of matrices:

Definition (Unitary/orthogonal matrices)

A matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ is **unitary** if $\mathbf{U}^* \mathbf{U} = \mathbf{I}_n$

A matrix $\mathbf{U} \in \mathbb{R}^{n \times n}$ is **orthogonal** if $\mathbf{U}^T \mathbf{U} = \mathbf{I}_n$

A unitary matrix is one whose columns are an orthonormal basis for $\mathbb{C}^n$ under the $\ell^2(\mathbb{C}^n)$ inner product.

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \qquad U^* U = \begin{pmatrix} \langle u_1, u_1 \rangle & \cdots & \langle u_1, u_n \rangle \\ \vdots & \ddots & \vdots \\ \langle u_n, u_1 \rangle & \cdots & \langle u_n, u_n \rangle \end{pmatrix}$$

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Theorem

Let $\mathbf{U} \in \mathbb{C}^{n \times n}$ be a unitary matrix. Then

- $\mathbf{U}^{-1} = \mathbf{U}^*$ ← Because $(U^*)U = I$
- $\mathbf{U}\mathbf{U}^* = \mathbf{I}$
- $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{C}^n$.

(i) $Ux = \sum_{j=1}^{n} x_j \underline{u}_j \rightarrow$ Pythagorean Thm

(ii) $\|Ux\|_2^2 = \langle Ux, Ux \rangle = (Ux)^* Ux = x^* U^* U x = \|x\|_2^2$

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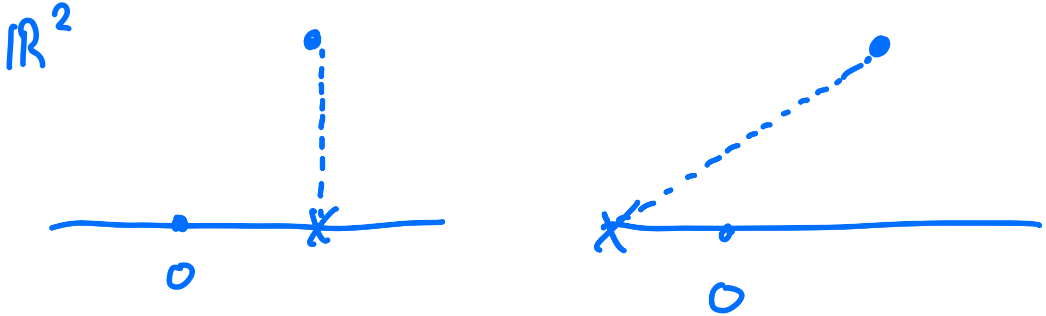
This last property in particular shows: unitary matrices are *isometries*, i.e., are ℓ^2- norm-preserving.
Unitary matrices have nice geometric interpretations: they are precisely rigid rotations and/or reflections.

Projections and projection matrices: Geometric construction

With $\mathbb{C}^n$ the ambient space, we want to define square matrix "projections". Informally, we want projections to

- Act like the identity on some subspace (the range)
- Annihilate components in another subspace (the kernel)



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In order to be well-defined, we need an additional condition.

Definition (Geometric definition of projection matrices)

A matrix $\mathbf{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if

1. $\mathbf{P}\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in V = \mathsf{range}(\mathbf{P})$
2. $\mathbf{P}\mathbf{w} = \mathbf{0}$ for all $\mathbf{w} \in W = \mathsf{ker}(\mathbf{P})$
3. $\mathsf{range}(\mathbf{P}) \oplus \mathsf{ker}(\mathbf{P}) = \mathbb{C}^n$ and $\mathsf{range}(\mathbf{P}) \cap \mathsf{ker}\,\mathbf{P} = \{0\}$. $\Longrightarrow n = \dim V + \dim W$

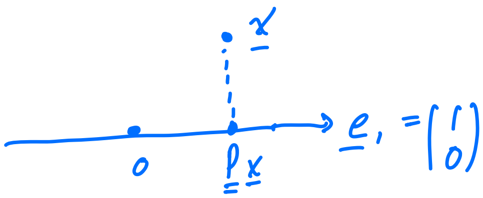
We typically say that $\mathbf{P}$ projects *onto* V, and projects *along* W.

Ex: $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$V = \mathrm{range}(P) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$, $W = \ker(P) = \mathrm{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$

$2 = \dim V + \dim W$

$V \cap W = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$, $\quad \mathbb{C}^2 = V \oplus W$



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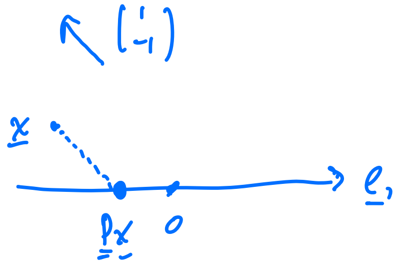
$V \oplus W = \mathbb{C}^2$, $\quad V \cap W = \{ \underline{0} \}$



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We typically say that $\mathbf{P}$ projects *onto* V, and projects *along* W.

This last condition is *necessary*:

- A projector $\mathbf{P}$ in $\mathbb{R}^3$ that projects onto $\mathsf{span}\{(1,0,0)\}$ along $\mathsf{span}\{(0,0,1)\}$ is undefined for input $(0,1,0)$.
- A projector $\mathbf{P}$ in $\mathbb{R}^2$ that projects onto $\mathbb{R}^2$ along $\mathsf{span}\{(1,0)\}$ is ill-defined.

Projections exist and are unique

Does this definition make sense?

Theorem

If V and W are $\mathbb{C}^N$-subspaces such that $V \cap W = \{\mathbf{0}\}$ and $\dim V + \dim W = N$, then $\exists$! projection matrix $\mathbf{P} \in \mathbb{C}^{N\times N}$ such that $\mathsf{range}(\mathbf{P}) = V$ and $\ker(\mathbf{P}) = W$.

(Pf: If P_1, P_2 are these projectors, show $P_1 v = P_2 v \quad \forall \underline{v} \in \mathbb{C}^N$)

How to construct projectors?

Initial computation: claim: Let $\underline{\underline{V}} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$, $\underline{\underline{W}} = \begin{pmatrix} | & & | \\ w_1 & \cdots & w_{N-n} \\ | & & | \end{pmatrix}$

be matrices whose columns hold a basis for the range, kernel, resp.

$[\underline{\underline{V}}\ \underline{\underline{W}}] \in \mathbb{C}^{N \times N}$ is invertible.

Claim: $[\underline{\underline{V}}\ \underline{\underline{W}}]^{-1}\underline{x}$ contains coordinates of x in the joint basis $\{\underline{v}_1 \dots \underline{V}_n, \underline{w}_1 \dots \underline{w}_{N-n}\}$.

(Why? $\underline{x} = \underline{v}_1$, what is $\underline{y} = [V\ W]^{-1}x$?

$$[\underline{V}\ \underline{\underline{W}}]\,\underline{y} = \underline{x} = \underline{v}_1$$

$$\Rightarrow \underline{y} = \underline{e}_1\ )$$

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This theorem ensures that any construction for $\mathbf{P}$ identifies the unique projector.

Let $n = \dim V$.
If $\mathbf{V} \in \mathbb{C}^{N\times n}$ satisfies $\mathsf{range}(\mathbf{V}) = V$, and $\mathbf{W} \in \mathbb{C}^{N\times(N-n)}$ satisfies $\mathsf{range}(\mathbf{W}) = W$, then

$$\mathbf{P} = \left[\mathbf{V}\ \ \mathbf{0}_{N\times(N-n)}\right]\left[\mathbf{V}\ \ \mathbf{W}\right]^{-1}$$

$$\left.\begin{array}{l}\text{(i) } \underline{\underline{P}}\,\underline{v} = \underline{v} \quad \forall\ \underline{v} \in V \\ \text{(ii) } \underline{\underline{P}}\,\underline{w} = \underline{0} \quad \forall\ \underline{w} \in W\end{array}\right\} \Rightarrow \underline{\underline{P}} \text{ is the unique proj. matrix.}$$

Notice: $I - P = [\underline{\underline{V}} \ \ \underline{\underline{W}}][\underline{\underline{V}} \ \ \underline{\underline{W}}]^{-1} - [\underline{\underline{V}} \ \ \underline{\underline{0}}][\underline{\underline{V}} \ \ \underline{\underline{W}}]^{-1}$

$= [\underline{\underline{0}} \ \ \underline{\underline{W}}][\underline{\underline{V}} \ \ \underline{\underline{W}}]^{-1}$

$\Rightarrow$ $I-P$ is the (unique) projector onto W along V.

Projection matrices: Algebraic definitions

There is a more algebraically convenient characterization of projection matrices.

A square matrix $\mathbf{A}$ is *idempotent* if $\mathbf{A} = \mathbf{A}^2$.

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Theorem

$\mathbf{P} \in \mathbb{C}^{n\times n}$ is a projection matrix iff it is idempotent.

Proof: Assume $P = P^2$

Given $x \in \mathbb{C}^n$, $x = \underbrace{Px}_{\in \operatorname{range}(P)} + \overbrace{(I-P)x}^{\in \ker(P)}$ since $P(I-P)x = (P-P^2)x = 0$

can $x \in \operatorname{range}(P)$ also lie in $\ker(P)$ if $x \neq 0$?

$0 \neq x = P y = P^2 y = P(Py) = Px$

This shows $\text{range}(P) \oplus \ker(P) = \mathbb{C}^n$

and $\text{range}(P) \cap \ker(P) = \{\underline{0}\}$.

Is $Px = x$ for $x \in \text{range}(P)$?

$x \in \text{range}(P) \implies x = Py = P^2 y = P(Py)$

$= Px$

"Yes"

Therefore: $P^2 = P \implies P$ is a projector.

Assume P is a projector. Let $x \in \mathbb{C}^n$ be arbitrary vector

$$x = Px + (I-P)x$$

$$Px = P^2 x + P\underbrace{(I-P)x}_{\in \ker(P)}$$

$$\implies Px = P^2 x \quad \forall\, x \implies P = P^2. \quad \blacksquare$$

Ex: $\underline{P_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\underline{P_2} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$P_1^2 = P_1$, $P_2^2 = P_2$

Projection matrices: Algebraic definitions

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A square matrix $\mathbf{A}$ is *idempotent* if $\mathbf{A} = \mathbf{A}^2$.

Theorem

$\mathbf{P} \in \mathbb{C}^{n \times n}$ is a projection matrix iff it is idempotent.

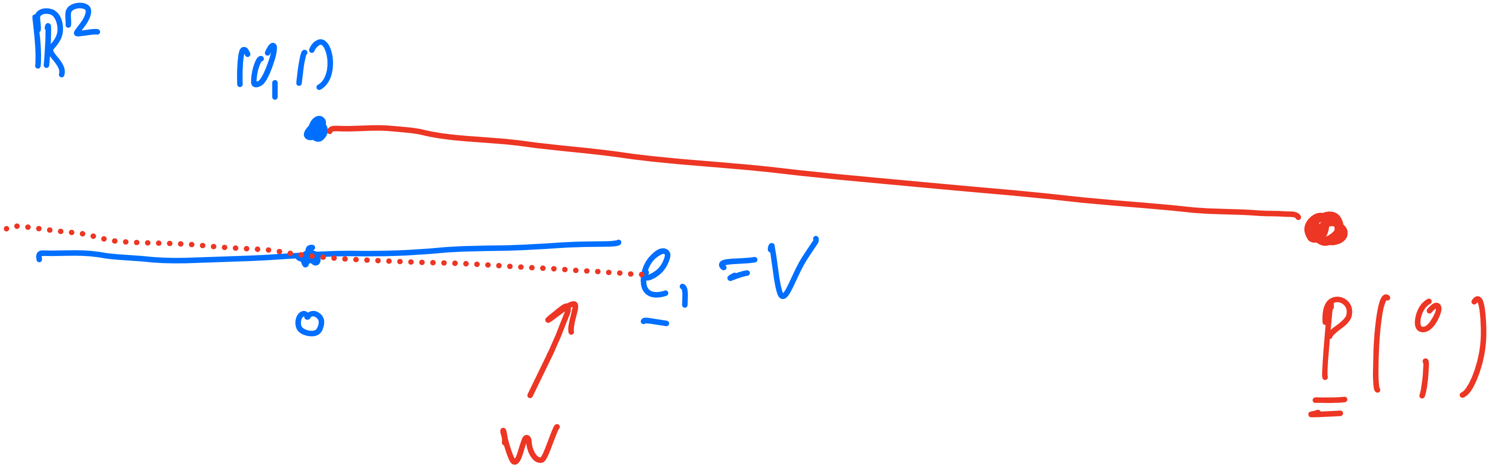
This motivates the more common definition of a projection matrix:

Definition (Algebraic definition of projection matrices)

$\mathbf{P} \in \mathbb{C}^{n \times n}$ is a projection matrix if $\mathbf{P} = \mathbf{P}^2$.

Geometry for orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.



Geometry for orthogonal projections

Projection matrices can in general inflate the size (norm) of non-trivial vectors by an arbitrary amount.

However, projecting along $\mathsf{range}(\mathbf{P})^\perp$ is a norm non-expansive operation, i.e.,

$$\mathsf{ker}(\mathbf{P}) = \mathsf{range}(\mathbf{P})^\perp \qquad \Longrightarrow \qquad \|\mathbf{P}\mathbf{v}\|_2 \leq \|\mathbf{v}\|_2.$$

$$\|v\|_2^2 = \|Pv + (I-P)v\|_2^2$$
$$= \|Pv\|_2^2 + \|(I-P)v\|_2^2$$
$$\geq \|Pv\|_2^2$$

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Projection matrices $\mathbf{P}$ satisfying $\mathsf{ker}(\mathbf{P}) = (\mathsf{range}\mathbf{P})^\perp$ are *orthogonal* projections.

Definition (Geometric definition of orthogonal projection matrices)

Let $\mathbf{P}$ be a projection matrix. It's an orthogonal projector if $\mathsf{ker}(\mathbf{P}) = \mathsf{range}(\mathbf{P})^\perp$.

If P is an orthogonal projector: choose basis for $\ker(P)$ and basis for $\text{range}(P)$ so that the joint basis is orthonormal.

$\Rightarrow$ $[\underline{\underline{V}} \;\; \underline{\underline{W}}]$ has orthonormal col's.

$$\underline{\underline{P}} = [\underline{\underline{V}} \;\; \underline{\underline{0}}][\underline{\underline{V}} \;\; \underline{\underline{W}}]^{-1} = [\underline{\underline{V}} \;\; \underline{\underline{0}}]\begin{bmatrix} \underline{\underline{V}}^* \\ \underline{\underline{W}}^* \end{bmatrix}$$

$$= \underline{\underline{V}}\,\underline{\underline{V}}^* + \underline{\underline{0}}\,\underline{\underline{W}}^*$$

$$\underline{\underline{P}} = \underline{\underline{V}}\,\underline{\underline{V}}^*$$

Orthogonal projectors

There is also an algebraically convenient characterization of orthogonal projection matrices.

Theorem

Let $\mathbf{P} \in \mathbb{C}^{n \times n}$ be a projection matrix. Then it is an orthogonal projector iff it is Hermitian.

$P = P^*$

Proof (half)

Assume P is an orthogonal projector (geometrically).

Take $x, y \in \mathbb{C}^n$, arbitrary.

$Px \in \text{range}(P)$

$(I-P)\cancel{x}\,y \in \ker(P)$

$\langle Px, (I-P)y \rangle = 0$

$y^*(I-P^*)Px = 0 \quad \forall\, x, y$

$\Rightarrow P - P^*P = 0$

$P = P^*P \qquad P^* = (P^*P)^* = P^*P$ ▨

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This motivates the more common definition of an orthogonal projection matrix:

Definition (Algebraic definition of orthogonal projectors)

$\mathbf{P} \in \mathbb{C}^{n\times n}$ is an orthogonal projection matrix if $\mathbf{P} = \mathbf{P}^2$ and $\mathbf{P} = \mathbf{P}^*$.

Permutation

A final class of matrices we'll consider is permutation matrices.

Definition

For a fixed $n \in \mathbb{N}$, $\pi : [n] \to [n]$ is a permutation if it is a bijection.

Every permutation of the set of indices of a vector corresponds to a linear operator on $\mathbb{C}^n$, so it can be encoded by an $n \times n$ matrix.

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Definition

$\mathbf{P} \in \mathbb{C}^{n \times n}$ is a permutation matrix if there is a permutation map π on $[n]$ such that $\mathbf{P}\mathbf{e}_j = \mathbf{e}_{\pi(j)}$ for all $j \in [n]$.

the jth column of P is $e_{\pi(j)}$

P's col's are a rearrangement of col's of I.

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Permutation matrices are orthogonal/unitary, and hence if $\mathbf{P}$ is a permutation matrix, then $\mathbf{P}^* = \mathbf{P}^{-1}$.

The space of permutation matrices is closed under matrix multiplication, so that if $\mathbf{P}$ and $\mathbf{Q}$ are permutation matrices, then so is $\mathbf{PQ}$.

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