Meta-Diagrams for 2-Parameter Persistence

Nate Clause[1], Tamal K. Dey[2*], Facundo Mémoli[1*], Bei Wang[3*]

[1*]Ohio State University, Columbus, OH, USA.
[2]Purdue University, West Lafayette, IN, USA.
[3]University of Utah, Salt Lake City, UT, USA.

*Corresponding author(s). E-mail(s): tamaldey@purdue.edu; facundo.memoli@gmail.com; beiwang@sci.utah.edu;
Contributing authors: nate.clause@gmail.com;

Abstract

We first introduce the notion of meta-rank for a 2-parameter persistence module, an invariant that captures the information behind images of morphisms between 1D slices of the module. We then define the meta-diagram of a 2-parameter persistence module to be the Möbius inversion of the meta-rank, resulting in a function that takes values from signed 1-parameter persistence modules. We show that the meta-rank and meta-diagram contain information equivalent to the rank invariant and the signed barcode. The equivalence leads to an algorithm for computing the meta-rank and meta-diagram of a 2-parameter module M indexed by a bifiltration of n simplices in $O(n^4)$ time. In addition, we define notions of erosion distance between meta-ranks and between meta-diagrams, and show that under these distances, meta-ranks and meta-diagrams are stable with respect to the interleaving distance. Lastly, the meta-diagram can be visualized in an intuitive fashion as a persistence diagram of diagrams, which generalizes the well-understood persistence diagram in the 1-parameter setting.

Keywords: Multiparameter persistence modules, persistent homology, Möbius inversion, barcodes, computational topology, topological data analysis

1 Introduction

In the case of a 1-parameter persistence module, the persistence diagram (or barcode) captures its complete information up to isomorphism via a collection of intervals. The persistence diagram is represented as a multi-set of points in the plane, whose coordinates are the birth and death times of intervals, each of which encodes the lifetime of a topological feature. This compact representation of a persistence module enables its interpretability and facilitates its visualization. When moving to the multiparameter setting, the situation becomes much more complex as a multiparameter persistence module may contain indecomposable pieces that are not entirely determined by intervals or do not admit a finite discrete description [1].

Such an increased complexity has led to the study of other invariants for multiparameter persistence modules. The first invariant is the *rank invariant* [1], which captures the information from the images of internal linear maps in a persistence module across all dimensions. Patel noticed that the persistence diagram in the 1-parameter setting is equivalent to the *Möbius inversion* [2] of the rank function [3]. He then defined the generalized persistence diagram as the Möbius inversion of a function defined on a subset of intervals of $\mathbb{R}$, denoted Dgm, with values in some abelian group.

The idea of Möbius inversion has been extended in many directions. Kim and Mémoli defined generalized persistence diagrams for modules on posets [4, 5]. Patel and McCleary extended Patel's generalized persistence diagrams to work for persistence modules indexed over finite lattices [6]. Botnan et al. [7] implicitly studied the Möbius inversion of the rank function for 2-parameter modules, leading to a notion of a diagram with domain all rectangles in $\mathbb{Z}^2$. Asashiba et al. used Möbius inversion on a finite 2D grid to define interval-decomposable approximations [8]. Morozov and Patel [9] defined a generalized persistence diagram in the 2-parameter setting via Möbius inversion of the birth-death function and provided an algorithm for its computation. Their algorithm has some similarity with ours: it utilizes the vineyards algorithm [10] to study a 2-parameter persistence module, by slicing it over 1D paths.

Our work also involves the idea of slicing a 2-parameter module. This idea of slicing appears in the fibered barcode [11, 12], which is equivalent to the rank function. To obtain insight into the structure of a 2-parameter persistence module M, Lesnick and Wright [12] explored a set of 1-parameter modules obtained via restricting M onto all possible lines of non-negative slope. Buchet and Escolar [13] showed that any 1-parameter persistence module with finite support could be found as a restriction of some indecomposable 2-parameter persistence module with finite support. Furthermore, Dey et al. [14] showed that certain zigzag (sub)modules of a 2-parameter module can be used to compute the generalized rank invariant, whose Möbius inversion is the generalized persistence diagram defined by Kim and Mémoli. Our work considers the images between slices of a 2-parameter module, which is related to the work by Bauer and Schmal [15].

In [16], Botnan et al. introduced the notion of *rank decomposition*, which is equivalent to the generalized persistence diagram formed by Möbius inversion of the rank function, under some additional conditions. Botnan et al. further demonstrated that the process of converting a module to a rank decomposition is stable with respect to the matching distance [17]. Additionally, they introduced a visualization of this rank decomposition via a *signed barcode*, which highlights the diagonals of rectangles appearing in the rank decomposition, along with their multiplicity. They visualized the value of the signed barcode with a 2-parameter persistence module generated by clustering a point cloud with a scale and a density parameter.

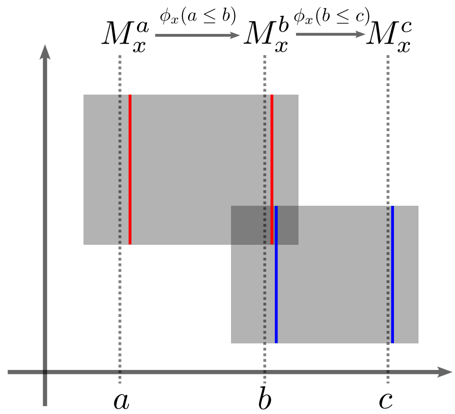


Fig. 1 Slicing a 2-parameter module M along vertical lines yields 1-parameter modules, such as M_x^a, M_x^b, and M_x^c. There are morphisms between these 1-parameter modules induced by the internal morphisms of M, and the meta-rank captures the information about these morphisms. For example, if M is defined as the direct sum of the two interval modules given by the two shaded rectangles, then the meta-rank of M on $[a, b)$ is the image of $\phi_x(a \leq b)$, which has a barcode consisting of the red interval. The meta-rank of M on $[b, c)$ has a barcode consisting of the blue interval, and the meta-rank of M on $[a, c)$ is 0, as $\phi_x(a \leq c) = \phi_x(b \leq c) \circ \phi_x(a \leq b) = 0$.

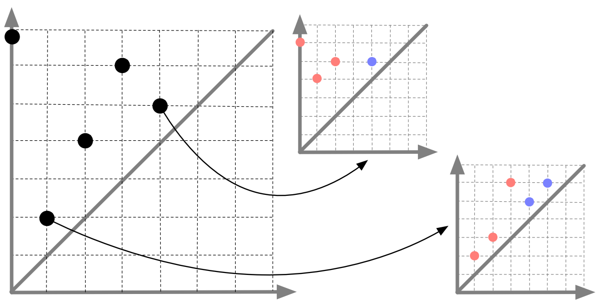


Fig. 2 A meta-diagram viewed as a persistence diagram of signed diagrams (red and blue mean positive and negative signs respectively).

Unlike the previous results that perform Möbius inversion over a higher-dimensional poset such as $\mathbb{Z}^2$, our work involves Möbius inversion over a finite subcollection of intervals of $\mathbb{R}$, which leads to a simpler inversion formula. In this work, we introduce the notion of *meta-rank* for a 2-parameter persistence module, which is a map from Dgm to isomorphism classes of persistence modules arising in a manner analogous to how the usual rank invariant is defined. Instead of looking at images of linear maps between vector spaces (as with the usual rank invariant), the meta-rank considers images of the maps between 1-parameter persistence modules formed by slicing a 2-parameter persistence module along vertical and horizontal lines, see Figure 1 and Remark 3.4. We then define the meta-diagram as the Möbius inversion of the meta-rank, giving a map from Dgm to isomorphism classes of signed persistence modules. This contrasts Botnan et al.'s approach [16] of using Möbius inversion in 2D, as our Möbius inversion formula over Dgm is simpler and consists of fewer terms.

Contributions. The meta-rank and meta-diagram turn out to contain information equivalent to the rank invariant (Proposition 3.8) and signed barcode (Proposition 4.11) respectively. Therefore, both meta-rank and meta-diagram can be regarded as these known invariants seen from a different perspective. Based on such equivalences, we introduce an algorithm for computing the meta-rank and meta-diagram of a 2-parameter module M indexed by a bifiltration of n simplices in $O(n^4)$ time[1]. However, this different viewpoint brings forth advantages as listed below that make the meta-rank and meta-diagram stand out on their own right:

1. The meta-diagram can be viewed as a persistence diagram of signed diagrams as illustrated in Figure 2. Such an intuitive visualization generalizes the classic persistence diagram – a known technique in topological data analysis (TDA) – to summarize persistent homology. For examples, see Section 5.2.
2. The meta-diagram also generalizes the concept of a sliced barcode well-known in TDA [12]. It assembles sliced bars on a set of lines, whilst remembering the maps between slices induced by the module M being sliced.

2 Preliminaries

We regard a poset $(P, \leq)$ as a category, with objects the elements $p \in P$, and a unique morphism $p \to q$ if and only if $p \leq q$; this is referred to as the *poset category* for $(P, \leq)$. When it is clear from the context, we will denote the poset category by P.

Fix a field $\mathbf{k}$, and assume all vector spaces have coefficients in $\mathbf{k}$ throughout this paper. Let $\mathbf{vec}$ denote the category of finite-dimensional vector spaces with linear maps between them. A *persistence module*, or *module* for short, is a functor $M : P \to \mathbf{vec}$. For any $p \in P$, we denote the vector space $M_p := M(p)$, and for any $p \leq q \in P$, we denote the linear map $\varphi_M(p \leq q) := M(p \leq q)$. When M is apparent, we drop the subscript from φ_M. We call P the *indexing poset* for M. We focus on the cases when the indexing poset is $\mathbb{R}$ or $\mathbb{R}^2$, equipped with the usual order and product order, respectively. Definitions and statements we make follow analogously when the indexing poset is $\mathbb{Z}$ or $\mathbb{Z}^2$, which we will cover briefly in Section 5. If the indexing poset for M is $P \subseteq \mathbb{R}$, then M is a *1-parameter (or 1D) persistence module*. If the indexing poset for M is $P \subseteq \mathbb{R}^2$, with P not totally-ordered, then M is a *2-parameter (or 2D) persistence module*, or a *bimodule* for short.

Following [18], we require that persistence modules be *constructible*:

Definition 2.1. Let $P \subset \mathbb{R}$. A module $M : P \to \mathbf{vec}$ is *constructible* if there exists a finite set $S := \{s_1 < \ldots < s_n\} \subset \mathbb{R}$ such that:

- If $a \in P$ with $a < s_1$, then $M(a) = 0$,
- If $a, b \in P$, then $M(a \leq b)$ is *not* an isomorphism only if there exists $1 \leq i \leq n$ such that either $a < s_i \leq b$ or $a \leq s_i < b$.

Let $P \subset \mathbb{R}^2$. Similarly to before, we say a bimodule $M : P \to \mathbf{vec}$ is *constructible* if there exists a finite set $S := \{s_1 < \ldots < s_n\} \subset \mathbb{R}$ such that:

[1] An earlier version of this paper, published at the 39th International Symposium on Computational Geometry (SoCG 2023), erroneously claimed an $O(n^3)$ algorithm, which we have corrected in this version.

- If $(x, y) \in P$ and $x < s_1$ or $y < s_1$, then $M((x, y)) = 0$,
- If $(x_1, y_1) \leq (x_2, y_2) \in P$, then $M((x_1, y_1) \leq (x_2, y_2))$ is *not* an isomorphism only if there exists $1 \leq i \leq n$ such that either $x_1 \leq s_i < x_2$, $x_1 < s_i \leq x_2$, $y_1 \leq s_i < x_2$, or $y_1 < s_i \leq y_2$.

In either case, such a module is *S-constructible.*

If a module is S-constructible, unless otherwise stated, assume $S = \{s_1 < \ldots < s_n\}$. If M is S-constructible, then M is S'-constructible for any $S' \supseteq S$. For the rest of the paper, we assume any given persistence module is constructible.

Of particular importance in the study of persistence modules are the notions of interval modules and interval decomposable modules. We state the following definitions:

Definition 2.2. For a poset $(P, \leq)$, an *interval* of P is a non-empty subset $I \subset P$ such that:

- (convexity) If $p, r \in I$ and $q \in P$ with $p \leq q \leq r$, then $q \in I$.
- (connectivity) For any $p, q \in I$, there is a sequence $p = r_0, r_1, \ldots, r_n = q$ of elements of I, where for all $0 \leq i \leq n-1$, either $r_i \geq r_{i+1}$ or $r_i \leq r_{i+1}$.

We denote the collection of all intervals of P as $\mathbf{Int}(P)$.

For $I \in \mathbf{Int}(P)$, the *interval module* $\mathbf{k}^I$ is the persistence module indexed over P, with:

$$\mathbf{k}^I_p = \begin{cases} \mathbf{k} & \text{if } p \in I \\ 0 & \text{otherwise} \end{cases}, \qquad \varphi_{\mathbf{k}^I}(p \leq q) = \begin{cases} \mathrm{id}_{\mathbf{k}} & \text{if } p \leq q \in I \\ 0 & \text{otherwise} \end{cases}.$$

Given any $M, N : P \to \mathbf{vec}$, the direct sum $M \oplus N$ is defined point-wise at each $p \in P$. We say a nontrivial $M : P \to \mathbf{vec}$ is *decomposable* if M is isomorphic to $N_1 \oplus N_2$ for some non-trivial $N_1, N_2 : P \to \mathbf{vec}$, which we denote by $M \cong N_1 \oplus N_2$. Otherwise, M is *indecomposable.* Interval modules are indecomposable [19].

A persistence module $M : P \to \mathbf{vec}$ is *interval decomposable* if it is isomorphic to a direct sum of interval modules. That is, if there is a multiset of intervals $\mathrm{barc}(M)$, such that:

$$M \cong \bigoplus_{I \in \mathrm{barc}(M)} \mathbf{k}^I$$

If this multiset exists, we call it the *barcode* of M. If it exists, $\mathrm{barc}(M)$ is well-defined as a result of the Azumaya-Krull-Remak-Schmidt theorem [20]. Thus, in the case where M is interval decomposable, $\mathrm{barc}(M)$ is a complete descriptor of the isomorphism type of M.

When $P = \mathbb{R}^2$, of particular importance in this work are *right-open rectangles*, which are intervals $R \subset \mathbb{R}^2$ of the form $R = [a_1, b_1) \times [a_2, b_2)$. If M can be decomposed as a direct sum of interval modules $\mathbf{k}^R$ with R a right-open rectangle, then we say M is *rectangle decomposable.*

1-parameter persistence modules are particularly nice, as they are always interval decomposable [21]. As a result, the barcode is a complete invariant for 1-parameter persistence modules. On the other hand, bimodules do not necessarily decompose in this way. In fact, there is no complete and simultaneously discrete descriptor for bimodules [1].

As a result, many necessarily incomplete invariants have been proposed to study bimodules. One notable such invariant is the *rank invariant* [1] recalled in Definition 2.3.

Definition 2.3 ([1])**.** For P a poset, define $\mathbf{D}(P) := \{(a,b) \in P \times P \,|\, a \leq b\}$. For $M : P \to \mathbf{vec}$, the *rank invariant of* M, $\mathrm{rank}_M : \mathbf{D}(P) \to \mathbb{Z}_{\geq 0}$, is defined point-wisely as:

$$\mathrm{rank}_M(a,b) := \mathrm{rank}(\varphi_M(a \leq b))$$

For a bimodule, the rank invariant is inherently a 4D object, making it difficult to visualize directly. RIVET [12] visualizes the rank invariant indirectly through the fibered barcode. In [16], Botnan et al. defined the *signed barcode* based on the notion of a *rank decomposition*:

Definition 2.4 ([16])**.** Let $M : \mathbb{R}^n \to \mathbf{vec}$ be a persistence module with rank function rank_M. Suppose $\mathscr{R}, \mathscr{S}$ are multisets of intervals from $\mathbb{R}^n$. Define $\mathbf{k}_{\mathscr{R}} := \oplus_{I \in \mathscr{R}} \mathbf{k}^R$, and similarly $\mathbf{k}_{\mathscr{S}}$. Then $(\mathscr{R}, \mathscr{S})$ is a *rank decomposition* for rank_M if as integral functions:

$$\mathrm{rank}_M = \mathrm{rank}_{\mathscr{R}} - \mathrm{rank}_{\mathscr{S}}.$$

If $\mathscr{R}, \mathscr{S}$ consist of right-open rectangles, then the pair is a rank decomposition by rectangles. We have:

Theorem 2.5 ([16], Theorem 3.3)**.** *Every finitely presented $M : \mathbb{R}^2 \to \mathbf{vec}$ admits a unique minimal rank decomposition by rectangles.*

Here minimality comes in the sense that $\mathscr{R} \cap \mathscr{S} = \emptyset$. The signed barcode then visualizes the rank function in $\mathbb{R}^2$ by showing the diagonals of the rectangles in $\mathscr{R}$ and $\mathscr{S}$.

3 Meta-Rank

In this section, we introduce the *meta-rank*. While the rank invariant captures the information of images between pairs of vector spaces in a persistence module, the meta-rank captures the information of images between two 1-parameter persistence modules obtained via slicing a bimodule.

We begin with some preliminary definitions:

Definition 3.1. Let $M : \mathbb{R}^2 \to \mathbf{vec}$ be a bimodule. For $s \in \mathbb{R}$, define the vertical slice $M_x^s : \mathbb{R} \to \mathbf{vec}$ point-wise as $M_x^s(a) := M(s,a)$, and with morphisms from a to b as $\varphi_x^s(a \leq b) := \varphi((s,a) \leq (s,b))$. Analogously, define the horizontal slice $M_y^s : \mathbb{R} \to \mathbf{vec}$ by setting $M_y^s(a) := M(a,s)$ and $\varphi_y^s(a \leq b) := \varphi((a,s) \leq (b,s))$ for all $a \leq b \in \mathbb{R}$.

Define a morphism of 1-parameter persistence modules $\phi_x(s \leq t) : M_x^s \to M_x^t$ for $s \leq t \in \mathbb{R}$ by $\phi_x(s \leq t)(a) := \varphi((s,a) \leq (t,a))$. Analogously, define $\phi_y(s \leq t) : M_y^s \to M_y^t$ for $s \leq t \in \mathbb{R}$ by $\phi_y(s \leq t)(a) := \varphi((a,s) \leq (a,t))$.

Denote by **Pvec** the collection of isomorphism classes of persistence modules over $\mathbb{R}$. Each element of **Pvec** can be uniquely represented by its barcode, which is what we do in practice. We recall the definition of Dgm from [3], which serves as the domain for the meta-rank:

Definition 3.2 ([3])**.** Define Dgm as the poset of all half-open intervals $[p,q) \subset \mathbb{R}$ for $p < q$, and all half-infinite intervals $[p,\infty) \subset \mathbb{R}$. The poset relation is inclusion.

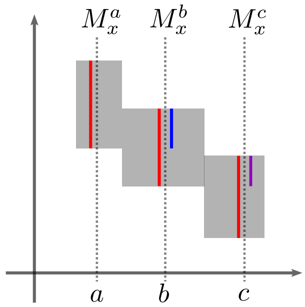


Fig. 3 An illustration of M and its barcode for some values of $\mathbf{mrk}_M$ in Example 3.5.

Definition 3.3. Suppose $M : \mathbb{R}^2 \to \mathbf{vec}$ is S-constructible. Define the *horizontal meta-rank* $\mathbf{mrk}_{M,x} : \mathrm{Dgm} \to \mathbf{Pvec}$ as follows:

- For $I = [s, s_i)$ with $s_i \in S$, $\mathbf{mrk}_{M,x}(I) := [\mathrm{im}(\phi_x(s \leq s_i - \delta))]$, for some $\delta > 0$ such that $s_i - \delta \geq s$ and $s_i - \delta \geq s_{i-1}$.
- For $I = [s, \infty)$, $\mathbf{mrk}_{M,x}(I) := [\mathrm{im}(\phi_x(s \leq s_n))]$.
- For all other $I = [s, t)$, $\mathbf{mrk}_{M,x}(I) := [\mathrm{im}(\phi_x(s \leq t))]$.

Analogously, define the *vertical meta-rank*, $\mathbf{mrk}_{M,y} : \mathrm{Dgm} \to \mathbf{Pvec}$ by replacing each instance of x above with y.

The results in this paper are stated in terms of the horizontal meta-rank, but hold analogously for the vertical meta-rank.

Remark 3.4. Note that, whereas the rank invariant assigns to every interval the isomorphism class (i.e. the dimension) of the image of the linear morphism induced by the interval, the meta-rank assigns to every interval the the isomorphism class of the persistence module corresponding to the image of the morphism of persistence modules induced by that interval. The latter isomorphism class is a barcode, which explains our definition.

To simplify notation, we henceforth denote $\mathbf{mrk}_{M,x}$ as $\mathbf{mrk}_M$. When there is no confusion, we drop the subscript from $\mathbf{mrk}_M$.

Example 3.5. As illustrated in Figure 3, let I be the connected gray interval and define the bimodule $M := \mathbf{k}^I$. The barcodes for the 1-parameter modules M_x^a, M_x^b, and M_x^c are shown in red next to their corresponding vertical slices. The barcode for $\mathbf{mrk}_M([a, b))$ consists of the blue interval, which is the overlap of the bars in M_x^a and M_x^b, $\mathrm{barc}(M_x^a) \cap \mathrm{barc}(M_x^b)$. Similarly, $\mathbf{mrk}_M([b, c))$ has a barcode consisting of the purple interval, which is the overlap of the bars in M_x^b and M_x^c. As the bars in the barcodes for M_x^a and M_x^c have no overlap, $\mathrm{im}(\phi_x(a \leq c)) = 0$, therefore $\mathbf{mrk}_M([a, c)) = 0$.

Remark 3.6. In general, $\mathbf{mrk}_x \neq \mathbf{mrk}_y$. For example, consider the right-open rectangle R with the lower-left corner the origin, and the upper right corner $(1, 2)$, as in Figure 4. Let $M := \mathbf{k}^R$. As illustrated, $\mathbf{mrk}_{M,x}([0, 1)) = [0, 2) \neq [0, 1) = \mathbf{mrk}_{M,y}([0, 1))$.

The following Proposition 3.7 allows us to compute the meta-rank of a bimodule via the meta-ranks of its indecomposable summands:

Proposition 3.7. *For $M, N : \mathbb{R}^2 \to \mathbf{vec}$, we have:*

$$\mathbf{mrk}_M \oplus \mathbf{mrk}_N = \mathbf{mrk}_{M \oplus N}$$

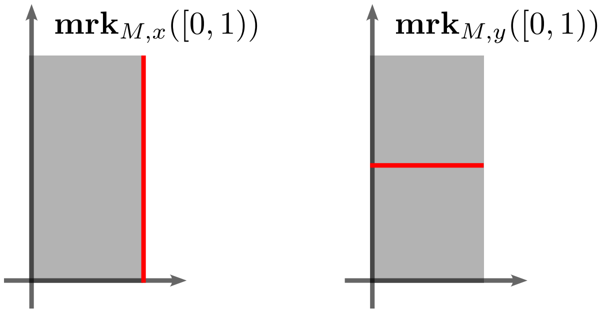


Fig. 4 An illustration of M, depicting $\mathbf{mrk}_{M,x}([0,1)) \neq \mathbf{mrk}_{M,y}([0,1))$.

where $\mathbf{mrk}_M \oplus \mathbf{mrk}_N : \mathrm{Dgm} \to \mathbf{Pvec}$ is defined as:

$$(\mathbf{mrk}_M \oplus \mathbf{mrk}_N)([s,t)) := [\mathbf{mrk}_M([s,t)) \oplus \mathbf{mrk}_N([s,t))].$$

For a finite $S \subseteq \mathbb{R}$, let $\overline{S} := S \cup \{\infty\}$. Define $\overline{S}_> : \mathbb{R} \cup \{\infty\} \to \overline{S}$ as $\overline{S}_>(t) := \min\left\{s \in \overline{S} \,|\, s > t\right\}$. For $M \in \mathbf{Pvec}$, $[b,d) \in \mathrm{Dgm}$, let $\#[b,d) \in M$ denote the multiplicity of $[b,d) \in \mathrm{barc}(M)$. The rank invariant and the meta-rank contain equivalent information:

Proposition 3.8. *For an S-constructible bimodule $M : \mathbb{R}^2 \to \mathbf{vec}$, one can compute* rank_M *from* $\mathbf{mrk}_M$ *and one can compute* $\mathbf{mrk}_M$ *from* rank_M*. In particular, given* $(s,y) \leq (t,y') \in \mathbb{R}^2$,

$$\mathrm{rank}_M((s,y),(t,y')) = \#\left\{[b_i,d_i) \in \mathbf{mrk}_M\left([s,\overline{S}_>(t))\right) \; s.t.\; b_i \leq y \leq y' < d_i\right\}.$$

That is, the rank is the number of intervals in $\mathrm{barc}\left(\mathbf{mrk}_M\left([s,\overline{S}_>(t))\right)\right)$ *containing* $[y,y']$.

The reason for needing $\overline{S}_>(t)$ for the right endpoint is that if $t \in S$, $\mathbf{mrk}_M([s,t))$ does not capture the information of the image of $\phi_x(s \leq t)$, only the image of $\phi_x(s \leq t-\delta)$.

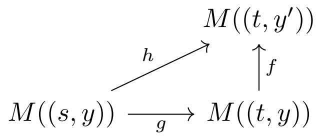
Proof. We start by showing that

$$\mathrm{rank}_M((s,y),(t,y')) = \#\{[b_i,d_i) \in \mathbf{mrk}_M\left([s,\overline{S}_>(t))\right) \; s.t.\; b_i \leq y \leq y' < d_i\}. \quad (1)$$

From the commutativity conditions on persistence modules, we have:

$$\varphi((s,y) \leq (t,y')) = \varphi((t,y) \leq (t,y')) \circ \varphi((s,y) \leq (t,y)),$$

and observe that $\varphi((s,y) \leq (t,y)) = \phi_x(s \leq t)|_{M((s,y))}$. From Definition 3.3, we have $\mathbf{mrk}_M\left([s,\overline{S}_>(t))\right) = [\mathrm{im}(\phi_x(s \leq t))]$. For simplified notations, let $h := \varphi((x,y) \leq (t,y'))$, $f := \varphi((t,y) \leq (t,y'))$, $g := \varphi((s,y) \leq (t,y))$, and $N := \mathrm{im}(\phi_x(s \leq t))$. We have a commutative diagram:



From elementary linear algebra, we have that $\text{rank}(h) = \text{rank}(g) - \dim(\ker f \cap \text{im}(g))$. As noted, $N(y) = \text{im}(g)$, so $\text{rank}(g) = \dim(N(y))$. It is immediate that $\dim(N(y))$ is equal to the number of intervals in $\text{barc}(N)$ which contains y. Furthermore, by the commutativity of internal morphisms of M, we have that $f|_{\text{im}(g)}$ is exactly the internal morphism $\varphi_N(y \leq y')$. From this and the rank-nullity theorem, we have:

$$\dim(N(y)) = \dim(\text{im}(\varphi_N(y \leq y'))) + \dim(\ker \varphi_N(y \leq y')).$$

As $\text{rank}(g) = \dim(N(y))$ and $\dim(\ker \varphi_N(y \leq y') = \dim(\ker f|_{\text{im}(g)}) = \dim(\ker f \cap \text{im}(g))$, we find $\text{rank}(h) = \dim(\text{im}(\varphi_N(y \leq y')))$. $\text{rank}(h)$ is precisely $\text{rank}_N(y \leq y')$, which is well-known to be the number of bars in $\text{barc}(N)$ containing $[y, y']$. As a result, we can compute rank_M from $\textbf{mrk}_M$.

Now we show the other claim, that we can compute $\textbf{mrk}_M$ from rank_M. By the definition of a constructible bimodule, there are finitely many constant regions for the rank function in its domain $\mathbf{D}(\mathbb{R})$. Taking the unions of the coordinates of the supremal points for each constant region results in a finite set S under which M is S-constructible. Hence, from rank_M we can find an S under which M is S-constructible.

Now fix a specific $S := \{s_1 < \ldots < s_n\}$ such that M is S-constructible. Fix some $[s, t) \in \text{Dgm}$, and fix an interval $[y, y') \in \text{Dgm}$. We show that from rank_M, we can determine the multiplicity of the interval $[y, y')$ in $\text{barc}(\textbf{mrk}_M([s, t)))$, denoted for conciseness as $\#[y, y')$. If $s < s_1$, then by Definition 3.2 we have $\textbf{mrk}_M([s, t)) = 0$. Thus, assume $s \geq s_1$ and define $S_<, S_\leq : \mathbb{R}_{\geq s_1} \cup \{\infty\} \to S$ by $S_<(t) := \max\{s \in S \mid s < t\}$ and $S_\leq(t) := \max\{s \in S \mid s \leq t\}$.

As a consequence of M being S-constructible, all intervals in $\text{barc}(\textbf{mrk}_M([s, t)))$ are of the form $[s_i, s_j)$ or $[s_i, \infty)$ for some $s_i, s_j \in S$. If $[y, y') = [s_i, s_j)$, then by the well-known inclusion-exclusion formula in 1-parameter persistence and the formula in Equation 1, we compute:

$$\begin{aligned}\#[s_i, s_j) = &\text{rank}_M((S_\leq(s), s_i), (S_<(t), s_{j-1})) - \text{rank}_M((S_\leq(s), s_i), (S_<(t), s_j)) \\ &+ \text{rank}_M((S_\leq(s), s_{i-1}), (S_<(t), s_j)) - \text{rank}_M((S_\leq(s), s_{i-1}), (S_<(t), s_{j-1})),\end{aligned} \tag{2}$$

where s_{n+1} is any value $s_{n+1} > s_n$, and s_0 is any value $s_0 < s_1$. If $[y, y') = [s_i, \infty)$, then analogously we compute:

$$\#[s_i, \infty) = \text{rank}_M((S_\leq(s), s_i), (S_<(t), s_n)) - \text{rank}_M((S_\leq(s), s_{i-1}), (S_<(t), s_n)).$$

Therefore, for any $[s, t) \in \text{Dgm}$, and $[y, y') \in \text{Dgm}$, we can compute the multiplicity of $[y, y') \in \textbf{mrk}_M([s, t))$ from rank_M, and so we can compute all of $\textbf{mrk}_M$ from rank_M. □

Finally, we discuss the stability of the meta-rank. The meta-rank is stable with respect to a notion of erosion distance, based on that of Patel [3]. Leading up to formulating the definition of our erosion distance, we define the notion of truncated barcode:

Definition 3.9. For $\epsilon \geq 0$, and $I = [s, t) \in \mathrm{Dgm}$, define $I[\epsilon :] := [s + \epsilon, t)$. For $M : \mathbb{R} \to \mathbf{vec}$ define the *truncated barcode*: $\mathrm{barc}_\epsilon(M) := \{I[\epsilon :] \,|\, I \in \mathrm{barc}(M)\}$. If $I = [s, t) \in \mathrm{barc}(M)$ has $t - s \leq \epsilon$, then I has no corresponding interval in $\mathrm{barc}_\epsilon(M)$.

Definition 3.10. For $M, N : \mathbb{R} \to \mathbf{vec}$, we say $M \preceq_\epsilon N$ if there exists an injective function on barcodes $\iota : \mathrm{barc}_\epsilon(M) \hookrightarrow \mathrm{barc}(N)$ such that for all $J \in \mathrm{barc}_\epsilon(M)$, $J \subseteq \iota(J)$.

For $\epsilon \geq 0$, $M \in \mathbf{Pvec}$, let M^ϵ refer to the ϵ-shift of M [22], with $M^\epsilon(a) := M(a+\epsilon)$ and $\varphi_{M^\epsilon}(a \leq b) := \varphi_M(a + \epsilon \leq b + \epsilon)$. For $I = [s, t) \in \mathrm{Dgm}$ and $a, b \in \mathbb{R}$, let $I_a^b := [s + a, t + b)$, with the convention $\infty + b := \infty$ for any $b \in \mathbb{R}$. We now define the erosion distance:

Definition 3.11. Let $M, N : \mathbb{R}^2 \to \mathbf{vec}$. Define the erosion distance as follows:

$$\begin{aligned}\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) := \inf\{\epsilon > 0 \,|\, \forall I \in \mathrm{Dgm}, \mathbf{mrk}_M(I_{-\epsilon}^{\epsilon})^\epsilon \preceq_{2\epsilon} \mathbf{mrk}_N(I) \text{ and} \\ \mathbf{mrk}_N(I_{-\epsilon}^{\epsilon})^\epsilon \preceq_{2\epsilon} \mathbf{mrk}_M(I)\}\end{aligned}$$

if the set we are infimizing over is empty, we set $\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) := \infty$.

Proposition 3.12. $\mathrm{d_E}$ *as defined in Definition 3.11 is an extended pseudometric on the collection of meta-ranks of constructible bimodules* $M : \mathbb{R}^2 \to \mathbf{vec}$.

We compare bimodules M and N using the multiparameter interleaving distance [22]. The ϵ-shift and the truncation of the barcode in Definition 3.9 are necessary for stability, due to the interleaving distance being based on diagonal shifts of bimodules, whereas the meta-rank is based on horizontal maps instead of diagonal ones. We have the following:

Theorem 3.13. *For constructible* $M, N : \mathbb{R}^2 \to \mathbf{vec}$*, we have:*

$$\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) \leq \mathrm{d_I}(M, N)$$

For the proofs of Proposition 3.12 and Theorem 3.13, see Appendix A.1.

4 Meta-Diagram

We use the Möbius inversion formula from Patel [3] on the meta-rank function to get a *meta-diagram*. This formula involves negative signs, so we need a notion of signed persistence modules. Our ideas are inspired by the work of Betthauser et al. [23], where we consider breaking a function into positive and negative parts.

Definition 4.1. A *signed 1-parameter persistence module* is an ordered pair (M, N), where $M, N : \mathbb{Z} \to \mathbf{vec}$ are 1-parameter persistence modules. M is the *positively signed* module, and N is the *negatively signed* module.

Definition 4.2. View $\mathbf{Pvec}$ as a commutative monoid with operation $\oplus$ given by $[M] \oplus [N] := [M \oplus N]$, and identity element $[0]$. Define $\mathbf{SPvec}$ to be the Grothendieck group of $\mathbf{Pvec}$.

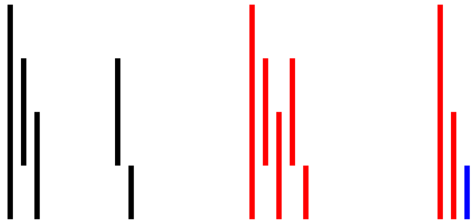


Fig. 5 Illustration of the barcodes for $M, N \in \mathbf{Pvec}$ and $M+N, M-N \in \mathbf{SPvec}$. For $M+N$ and $M-N$, a red interval is positively signed and a blue interval is negatively signed.

Each element of **SPvec** is an isomorphism class of ordered pairs $[([M^+], [M^-])]$. From the completeness of barcodes for 1-parameter persistence modules, we assume without loss of generality that $M^{\pm} := \oplus_{I \in \mathrm{barc}(M^{\pm})} \mathbf{k}^I$ and drop the internal equivalence class notation to write an element of **SPvec** as $[(M^+, M^-)]$. Proposition 4.3 allows us to make a canonical choice of representative for each element of **SPvec**:

Proposition 4.3. *Let* $A \in \mathbf{SPvec}$*. Then there is a unique representative* $A = [(M^+, M^-)]$ *with* $\mathrm{barc}(M^+) \cap \mathrm{barc}(M^-) = \emptyset$*.*

Proof. First, we establish the existence of such a pair (M^+, M^-). Suppose $A \in \mathbf{SPvec}$ has a representative (M_1^+, M_1^-), with $\mathcal{J} := \mathrm{barc}(M_1^+) \cap \mathrm{barc}(M_1^-) \neq \emptyset$. Define $M_2^+ := \oplus_{I \in \mathrm{barc}(M_1^+) \setminus \mathcal{J}} \mathbf{k}^I$, $M_2^- := \oplus_{I \in \mathrm{barc}(M_1^-) \setminus \mathcal{J}} \mathbf{k}^I$, and $V := \oplus_{I \in \mathcal{J}} \mathbf{k}^I$. Consider $M_1^+ \oplus M_2^- \oplus V$ and $M_2^+ \oplus M_1^- \oplus V$. By construction, both of these have barcode $\mathrm{barc}(M_1^+) \cup \mathrm{barc}(M_1^-)$, where $\cup$ is the multiset union. Hence, these two modules are isomorphic. As a result, in **SPvec**, we have $[(M_1^+, M_1^-)] = [(M_2^+, M_2^-)]$, and by construction (M_2^+, M_2^-) is a representative with $\mathrm{barc}(M_2^+) \cap \mathrm{barc}(M_2^-) = \emptyset$.

Now we establish the uniqueness of the pair (M^+, M^-). Suppose that $[(M_1^+, M_1^-)] = [(M_2^+, M_2^-)]$, $\mathrm{barc}(M_1^+) \cap \mathrm{barc}(M_1^-) = \emptyset$, and $\mathrm{barc}(M_2^+) \cap \mathrm{barc}(M_2^-) = \emptyset$. It is a simple algebraic fact that for two 1-parameter persistence modules M and N, $\mathrm{barc}(M \oplus N) = \mathrm{barc}(M) \cup \mathrm{barc}(N)$, where $\cup$ is the multiset union. By definition of $[(M_1^+, M_1^-)] = [(M_2^+, M_2^-)]$, there must exist a 1-parameter persistence module V such that $M_1^+ \oplus M_2^- \oplus V \cong M_2^+ \oplus M_1^- \oplus V$. This implies that $\mathrm{barc}(M_1^+) \cup \mathrm{barc}(M_2^-) = \mathrm{barc}(M_2^+) \cup \mathrm{barc}(M_1^-)$. By our assumptions on intersections, this implies that $\mathrm{barc}(M_1^+) = \mathrm{barc}(M_2^+)$ and $\mathrm{barc}(M_1^-) = \mathrm{barc}(M_2^-)$, which means $(M_1^+, M_1^-) = (M_2^+, M_2^-)$. Therefore, this is the unique representative satisfying our intersection criterion. □

As a result of Proposition 4.3, when convenient, we represent an element of **SPvec** uniquely by the sum of barcodes of this special representative, as in the following example:

Example 4.4. Consider $[(N^+, N^-)] \in \mathbf{SPvec}$ where $\mathrm{barc}(N^+) = \{[0,4],[1,3],[2,4]\}$ and $2\mathrm{barc}(N^-) = \{[1,3],[3,4]\}$. By Proposition 4.3, $[(N^+, N^-)]$ is uniquely represented by $[(M^+, M^-)]$ with $\mathrm{barc}(M^+) = \{[0,4],[2,4]\}$ and $\mathrm{barc}(M^-) = \{[3,4]\}$. In practice, we will denote this element of **SPvec** as $[0,4] + [2,4] - [3,4] \in \mathbf{SPvec}$. If $M, N \in \mathbf{Pvec}$, denote by $M + N$ the element $[(M \oplus N, 0)] \in \mathbf{SPvec}$, and denote by $M - N$ the element $[(M, N)] \in \mathbf{SPvec}$. For an illustration, see Figure 5.

With this notion of signed persistence module in hand, we now use a modified version of the Möbius inversion formula from [3] to define a meta-diagram:

Definition 4.5. Let $M : \mathbb{R}^2 \to \mathbf{vec}$ be S-constructible. Define the *horizontal meta-diagram* to be the function $\mathbf{mdgm}_M : \mathrm{Dgm} \to \mathbf{SPvec}$ via the Möbius inversion formula:

$$\begin{aligned}\mathbf{mdgm}_{M,x}([s_i, s_j)) &:= \mathbf{mrk}_{M,x}([s_i, s_j)) - \mathbf{mrk}_{M,x}([s_i, s_{j+1})) \\ &\quad + \mathbf{mrk}_{M,x}([s_{i-1}, s_{j+1})) - \mathbf{mrk}_{M,x}([s_{i-1}, s_j)) \\ \mathbf{mdgm}_{M,x}([s_i, \infty)) &:= \mathbf{mrk}_{M,x}([s_i, \infty)) - \mathbf{mrk}_{M,x}([s_{i-1}, \infty))\end{aligned}$$

where s_0 is any value $s_0 < s_1$ and s_{n+1} is any value $s_{n+1} > s_n$. For any other $[s,t) \in \mathrm{Dgm}$, set $\mathbf{mdgm}_{M,x}([s,t)) := 0$. Define the *vertical meta-diagram* by replacing each instance of x above with y.

We henceforth let **mdgm** refer to the horizontal meta-diagram of M, dropping the subscript when there is no confusion. The following Möbius inversion formula describes the relation between the meta-rank and meta-diagram. It is the direct analogue of [3, Theorem 4.1].

Proposition 4.6. *For* $[s,t) \in \mathrm{Dgm}$*, we have:*

$$\mathbf{mrk}([s,t)) = \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [s,t)}} \mathbf{mdgm}(I)$$

Proof. Suppose $s = s_i < t = s_j$. Then we have:

$$\begin{aligned}\sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [s,t)}} \mathbf{mdgm}(I) &= \sum_{k=j}^{n} \sum_{h=1}^{i} \mathbf{mdgm}([s_h, s_k)) + \sum_{h=1}^{i} \mathbf{mdgm}([s_h, \infty)) \\ &= \sum_{k=j}^{n} \sum_{h=1}^{i} \big(\mathbf{mrk}([s_h, s_k)) - \mathbf{mrk}([s_h, s_{k+1})) \\ &\qquad + \mathbf{mrk}([s_{h-1}, s_{k+1})) - \mathbf{mrk}([s_{h-1}, s_k))\big) \\ &\quad + \sum_{h=1}^{i} \big(\mathbf{mrk}([s_h, \infty)) - \mathbf{mrk}([s_{h-1}, \infty))\big) \\ &= \sum_{k=j}^{n} \big(\mathbf{mrk}([s_i, s_k)) - \mathbf{mrk}([s_i, s_{k+1}))\big) + \mathbf{mrk}([s_i, \infty)) \\ &= \mathbf{mrk}([s_i, s_j))\end{aligned}$$

Now suppose $s = s_i < t = \infty$. We have:

$$\begin{aligned}\sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [s,t)}} \mathbf{mdgm}([s_i, \infty)) &= \sum_{h=1}^{i} \mathbf{mdgm}([s_h, \infty)) \\ &= \sum_{h=1}^{i} \big(\mathbf{mrk}([s_h, \infty)) - \mathbf{mrk}([s_{h-1}, \infty))\big) \\ &= \mathbf{mrk}([s_i, \infty))\end{aligned}$$

□

Proposition 4.7. *For $M, N : \mathbb{R}^2 \to \mathbf{vec}$, we have:*

$$\mathbf{mdgm}_M \oplus \mathbf{mdgm}_N = \mathbf{mdgm}_{M \oplus N},$$

where $\mathbf{mdgm}_M \oplus \mathbf{mdgm}_N : \mathrm{Dgm} \to \mathbf{SPvec}$ *is defined by*

$$\begin{aligned}(\mathbf{mdgm}_M \oplus \mathbf{mdgm}_N)([s,t)) := [&\mathbf{mdgm}_M([s,t))^+ \oplus \mathbf{mdgm}_N([s,t))^+, \\ &\mathbf{mdgm}_M([s,t))^- \oplus \mathbf{mdgm}_N([s,t))^-].\end{aligned}$$

Proposition 4.7 allows us to compute meta-diagrams straightforwardly if we have an indecomposable decomposition of a module. In particular, by Proposition 4.8 and Corollary 4.9, meta-diagrams are simply computable for rectangle decomposable modules.

Proposition 4.8. *Suppose $M = \mathbf{k}^R$ is an $\mathbb{R}^2$-indexed interval module supported on the right-open rectangle R, with lower-left corner (s,t) and upper-right corner (s', t'). We have:*

$$\mathbf{mdgm}_M([a,b)) = \begin{cases} [t, t') & \text{if } a = s \text{ and } b = s'; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First, note that M is constructible, over some set $S = \{s_1 < \ldots < s_4\}$ of size no more (but potentially less than) three, with S consisting of s, s', t, and t'. It is straightforward to compute the following:

$$\mathbf{mrk}([a,b)) = \begin{cases} [t, t') & \text{if } s \leq a \leq b < s'; \\ 0 & \text{otherwise,} \end{cases}$$

as $\mathrm{im}(a \leq b)$ is either the image of $[t, t')$ under the identity, or trivial.

Assume without loss of generality that $s = s_a$ and $s' = s_b$. If $a, b \notin S \times S$, then immediately $\mathbf{mdgm}([a,b)) = 0$ by definition. To compute the remainder of the meta-diagram, for each pair $s_i < s_j$, we need to compute the four meta-ranks $\mathbf{mrk}([s_i, s_j))$, $\mathbf{mrk}([s_i, s_{j+1}))$, $\mathbf{mrk}([s_{i-1}, s_{j+1}))$, and $\mathbf{mrk}([s_{i-1}, s_j))$. We now break into cases based on where s_i, s_j are, the domains and codomains of the image maps ϕ in the meta-rank definition:

- Case 1: $s_i < s$. All four meta-ranks are trivial since the domains M_{s_i}, $M_{s_{i-1}}$ are trivial modules. Hence, $\mathbf{mdgm}([s_i, s_j)) = 0$.
- Case 2: $s_j > s'$. All four meta-ranks are trivial since the codomains M_{s_j}, $M_{s_{j+1}}$ are trivial modules, and $\mathbf{mdgm}([s_i, s_j)) = 0$.
- Case 3: $s_i = s$, $s < s_j < s'$. We have $\mathbf{mrk}([s_i, s_j)) = [t, t')$, $\mathbf{mrk}([s_i, s_{j+1})) = [t, t')$, $\mathbf{mrk}([s_{i-1}, s_{j+1}) = 0$ and $\mathbf{mrk}([s_{i-1}, s_j)) = 0$, so $\mathbf{mdgm}([s_i, s_j)) = 0$.
- Case 4: $s < s_i < s', s_j = s'$. We have $\mathbf{mrk}([s_i, s_j)) = [t, t')$, $\mathbf{mrk}([s_i, s_{j+1})) = 0$, $\mathbf{mrk}([s_{i-1}, s_{j+1})) = 0$ and $\mathbf{mrk}([s_{i-1}, s_j)) = [t, t')$, so $\mathbf{mdgm}([a, b]) = 0$.
- Case 5: $s < s_i < s_j < s'$. All four meta-ranks are $[t, t')$, so $\mathbf{mdgm}([s_i, s_j)) = 0$.
- Case 6: $s_i = s$, $s_j = s'$. We have $\mathbf{mrk}([s_i, s_j)) = [t, t')$, $\mathbf{mrk}([s_i, s_{j+1})) = 0$, $\mathbf{mrk}([s_{i-1}, s_{j+1})) = 0$ and $\mathbf{mrk}([s_i, s_{j+1})) = 0$, so $\mathbf{mdgm}([s_i, s_j)) = [t, t')$.

This exhausts the cases for positions of s_i and s_j relative to s and s', and so we are done. □

Corollary 4.9. *Let $M = \oplus_{R \in \mathrm{barc}(M)} \mathbf{k}^R$ be rectangle decomposable. Then the interval $[t, t')$ appears in $\mathbf{mdgm}([s, s'))$ with multiplicity n if and only if the right-open rectangle with lower-left corner (s, t) and upper right corner (s', t') appears in* barc(M) *with multiplicity n.*

Remark 4.10. In the classical persistence diagram, a point (b, d) above the diagonal corresponds to a persistent feature which is born at b and dies at d. The meta-diagram can be visualized as a "diagram of diagrams" as illustrated in Figure 2 and exemplified in Section 5.2. We refer to the diagram on the left of Figure 2 with which we select an interval to evaluate a meta-diagram as the "outer diagram" and the diagram(s) on the right of a meta-diagram evaluated at a specific interval(s) in the outer diagram as an "inner diagram". In light of Proposition 4.8 and Corollary 4.9, we offer the following intuition for points in these diagrams. A point (b, d) in the outer diagram corresponds to a persistent feature which is born at b and dies at d *with respect to the x direction*, and a point (b, d) in an inner diagram corresponds to a persistent feature which is born at b and dies at d *with respect to the y direction.*

4.1 Equivalence With Rank Decomposition via Rectangles

For $M : \mathbb{R}^2 \to \mathbf{vec}$, the rank decomposition by rectangles contains the same information as the rank invariant, which by Proposition 3.8 contains the same information as the meta-rank. We now show one can directly go from the meta-diagram to the rank decomposition:

Proposition 4.11. *Let $M : \mathbb{R}^2 \to \mathbf{vec}$ be constructible. Define as follows:*

$$\mathscr{R} := \bigcup_{I \in \mathrm{Dgm}} \left(\bigcup_{[a,b) \in \mathbf{mdgm}_M(I)} I \times [a, b) \right),$$

$$\mathscr{S} := \bigcup_{I \in \mathrm{Dgm}} \left(\bigcup_{-[a,b) \in \mathbf{mdgm}_M(I)} I \times [a, b) \right),$$

where all unions are the multiset union. Then $(\mathscr{R}, \mathscr{S})$ is a rank decomposition for M.

Proof. It suffices to show that for all $w_1 := (x_1, y_1) \leq w_2 := (x_2, y_2) \in \mathbb{R}^2$, $\mathrm{rank}_M(w_1, w_2) = \mathrm{rank}_{\mathbf{k}_{\mathscr{R}}}(w_1, w_2) - \mathrm{rank}_{\mathbf{k}_{\mathscr{S}}}(w_1, w_2)$. Suppose $w_1 \leq w_2 \in \mathbb{R}^2$ as above. By Proposition 3.8,

$$\mathrm{rank}_M(w_1, w_2) = \#\{[b_i, d_i) \in \mathbf{mrk}_M([x_1, x_2')) \ \text{ s.t. } \ b_i \leq y_1 \leq y_2 < d_i\},$$

where for notational simplicity, $x_2' := \overline{S}_>(x_2)$.

Now fix $[b, d)$ such that $b \leq y_1 \leq y_2 < d$. By Proposition 4.6, we have:

$$\#\{[b,d) \in \mathbf{mrk}_M([x_1, x_2'))\} = \#\left\{[b,d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2')}} \mathbf{mdgm}_M(I)\right\}$$

$$= \left(\#\left\{[b,d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2')}} \mathbf{mdgm}_M^+(I)\right\}\right) - \left(\#\left\{[b,d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2')}} \mathbf{mdgm}_M^-(I)\right\}\right)$$

By Proposition 4.8 and Corollary 4.9, the term $\#\left\{[b,d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2')}} \mathbf{mdgm}^+(I)\right\}$ is the number of times $I \times [b, d)$ appears in $\mathscr{R}$ across all $I \supseteq [x_1, x_2')$, and the term $\#\left\{[b,d) \in \sum_{\substack{I \in \mathrm{Dgm} \\ I \supseteq [x_1, x_2')}} \mathbf{mdgm}^-(I)\right\}$ is the number of times $I \times [b, d)$ appears in $\mathscr{S}$ across all $I \supseteq [x_1, x_2')$.

Thus, we see that $\mathrm{rank}_M(w_1, w_2)$ is equal to the number of rectangles in $\mathscr{R}$ containing w_1 and w_2 minus the number of rectangles in $\mathscr{S}$ containing w_1 and w_2. From the definition of rectangle module and the fact that rank commutes with direct sums, the first term is $\mathrm{rank}(\mathbf{k}_{\mathscr{R}})(w_1, w_2)$ and the second term is $\mathrm{rank}(\mathbf{k}_{\mathscr{S}})(w_1, w_2)$, and so we get:

$$\mathrm{rank}_M(w_1, w_2) = \mathrm{rank}_{\mathbf{k}_{\mathscr{R}}}(w_1, w_2) - \mathrm{rank}_{\mathbf{k}_{\mathscr{S}}}(w_1, w_2) \qquad \square$$

4.2 Stability of Meta-Diagrams

We now state a stability result for meta-diagrams. We need to modify the notion of erosion distance to do so, as meta-diagrams have negatively signed parts. We proceed by adding the positive part of one meta-diagram to the negative part of the other. This idea stems from Betthauser et al.'s work [23], and was also used in the stability of rank decompositions in [16].

Definition 4.12. For $M, N : \mathbb{R}^2 \to \mathbf{vec}$, define $\mathrm{PN}(M, N) : \mathrm{Dgm} \to \mathbf{vec}$ as

$$\mathrm{PN}(M, N)([s, t)) := \mathbf{mdgm}_M^+([s, t)) + \mathbf{mdgm}_N^-([s, t))$$

$\mathrm{PN}(M,N)([s,t))$ is a non-negatively signed 1-parameter persistence module for all $[s,t) \in \mathrm{Dgm}$, allowing us to make use of the previous notion of $\preceq_\epsilon$ (Definition 3.10) to define an erosion distance for meta-diagrams. Unlike meta-ranks which have a continuous support, a meta-diagram is only supported on $\left(\overline{S}\right)^2$ for some finite $S \subset \mathbb{R}$. As a result, we first modify the notion of erosion distance to fit the discrete setting.

Define maps $\overline{S}_{\geq}, \overline{S}_{\leq} : \mathbb{R} \cup \{\infty\} \to \overline{S}$ by $\overline{S}_{\geq}(x) := \min\left\{s \in \overline{S} \,|\, x \geq s\right\}$ and $\overline{S}_{\leq}(x) := \max\left\{s \in \overline{S} \,|\, x \leq s\right\}$, or some value less than s_1 if this set is empty. We say S is *evenly-spaced* if there exists $c \in \mathbb{R}$ such that $s_{i+1} - s_i = c$ for all $1 \leq i \leq n-1$. In the following, fix an evenly-spaced finite $S \subset \mathbb{R}$.

Definition 4.13. For S-constructible $M, N : \mathbb{R}^2 \to \mathbf{vec}$, define the erosion distance:

$$\begin{aligned}\mathrm{d}_{\mathrm{E}}^S(\mathbf{mdgm}_M, \mathbf{mdgm}_N) := \inf\{\epsilon \geq 0 \,|\, &\forall s \leq t \in \overline{S},\\ &\mathrm{PN}(M,N)([\overline{S}_{\leq}(s-\epsilon), \overline{S}_{\geq}(s+\epsilon))^\epsilon \preceq_{2\epsilon} \mathrm{PN}(N,M)([s,t)) \text{ and}\\ &\mathrm{PN}(N,M)([\overline{S}_{\leq}(s-\epsilon), \overline{S}_{\geq}(s+\epsilon))^\epsilon \preceq_{2\epsilon} \mathrm{PN}(M,N)([s,t))\}\end{aligned}$$

Proposition 4.14 follows analogously to Proposition 3.12:

Proposition 4.14. *$\mathrm{d}_{\mathrm{E}}^S$ as defined in Definition 4.13 is an extended pseudometric on the collection of meta-diagrams of S-constructible modules $M : \mathbb{R}^2 \to \mathbf{vec}$.*

We have the following stability result for meta-diagrams,

Theorem 4.15. *For S-constructible $M, N : \mathbb{R}^2 \to \mathbf{vec}$, with S evenly-spaced, we have*

$$\mathrm{d}_{\mathrm{E}}^S(\mathbf{mdgm}_M, \mathbf{mdgm}_N) \leq \mathrm{d}_{\mathrm{I}}(M,N).$$

For the proof of Theorem 4.15, as well as a more general stability result when S is not evenly-spaced, see Appendix A.2.

5 Algorithm

In this section, we provide an algorithm to compute the meta-rank and meta-diagram for a module over a finite grid in $\mathbb{Z}^2$. The input to this algorithm is a simplex-wise bifiltration:

Definition 5.1. Let $n \in \mathbb{Z}$, and $[n]$ denote the poset $\{1, \ldots, n\}$ with the usual order. Let K be a simplicial complex, and $\mathrm{sub}(K)$ denote all subsets of K which are themselves simplicial complexes. A filtration is a function $F : [n] \to \mathrm{sub}(K)$ such that for $a \leq b$, $F(a) \subseteq F(b)$. We say a filtration is simplex-wise if for all $1 \leq a \leq n-1$, either $F(a+1) = F(a)$ or $F(a+1) = F(a) \cup \{\sigma\}$ for some $\sigma \in K \setminus F(a)$. In the latter case, we denote this with $F(a) \xrightarrow{\sigma} F(a+1)$. We say $\sigma \in \mathrm{sub}(K)$ arrives at a if $\sigma \in F(a)$ and $\sigma \notin F(a-1)$.

Define $P_n := [n] \times [n]$ equipped with the product order. A bifiltration is a function $F : P_n \to \mathrm{sub}(K)$. We say a bifiltration is simplex-wise if for all $(a,b) \in P_n$, for $(x,y) = (a+1,b)$ or $(a,b+1)$, if $(x,y) \in P_n$, then either $F((x,y)) = F((a,b))$, or $F((a,b)) \xrightarrow{\sigma} F((x,y))$ for some $\sigma \notin F((a,b))$.

Applying homology to a bifiltration yields a bimodule defined on P_n. Our theoretical background in previous sections focused on the case of bimodules defined over $\mathbb{R}^2$.

The same ideas and major results follow similarly for a module defined over P_n. We quickly highlight the differences in definitions when working with modules defined on P_n. The following definitions are re-phrasings of the horizontal meta-rank and horizontal meta-diagram for modules indexed over P_n, but as before, the statements are directly analogous in the vertical setting. Let $\mathbf{Int}([n])$ refer to all intervals of $[n]$, which consists of $\{[a,b] \mid a \leq b,\ a,b \in [n]\}$.

Definition 5.2. For $M : P_n \to \mathbf{vec}$, define the meta-rank, $\mathbf{mrk}_M : \mathbf{Int}([n]) \to \mathbf{Pvec}$ by

$$\mathbf{mrk}_M([s,t]) := [\mathrm{im}(\phi_x(s \leq t))]$$

Definition 5.3. For $M : P_n \to \mathbf{vec}$, define the meta-diagram, $\mathbf{mdgm}_M : \mathbf{Int}([n]) \to \mathbf{SPvec}$ as follows: if $1 < s \leq t < n$, define:

$$\begin{aligned}
\mathbf{mdgm}_M([s,t]) &:= \mathbf{mrk}_M([s,t]) - \mathbf{mrk}_M([s,t+1]) \\
&\qquad + \mathbf{mrk}_M([s-1,t+1]) - \mathbf{mrk}_M([s-1,t]), \\
\mathbf{mdgm}_M([s,n]) &:= \mathbf{mrk}_M([s,n]) - \mathbf{mrk}_M([s-1,n]), \\
\mathbf{mdgm}_M([1,t]) &:= \mathbf{mrk}_M([1,t]) - \mathbf{mrk}_M([1,t+1]), \text{ and} \\
\mathbf{mdgm}_M([1,n]) &:= \mathbf{mrk}_M([1,n]).
\end{aligned}$$

5.1 Overview of the Algorithm

Henceforth, assume $F : P_n \to \mathrm{sub}(K)$ is a simplex-wise bifiltration. The input to our algorithm is such an F, and the output is the meta-rank and meta-diagram of the bimodule bimodule $M := \oplus_{k\geq 0} H_k(F)$. The output meta-rank and meta-diagram are stored as a *list of lists*, where an index in the outer list corresponds to an interval $[s,t] \in \mathbf{Int}([n])$, and an inner list consists of all intervals in the (signed) persistence diagram corresponding to the meta-rank or meta-diagram evaluated at a specific $[s,t] \in \mathbf{Int}([n])$.

Step one of our algorithm is to compute the rank invariant of M. We can do so using existing algorithms [16, 24] which rely on the vineyards approach [10] to compute ranks between all pairs of points $a \leq b \in P_n$ by sweeping through a collection of paths, at least one of which must contain a and b. This step is well-known to have a time complexity $O(n^4)$.

Step two of our algorithm iterates over all pairs $[s,t], [a,b] \in \mathbf{Int}([n])$. For each pair $[s,t], [a,b]$, we compute the cardinality of $[a,b] \in \mathbf{mrk}_M([s,t])$ directly via Equation 2. With the above notation, there are $O(n^2)$ possible intervals $[s,t]$, and for a fixed $[s,t]$ there are $O(n^2)$ possible intervals $[a,b]$. Computing the cardinality of $[a,b] \in \mathbf{mrk}_M([s,t])$ via Equation 2 is $O(1)$, so the complexity of Step two is $O(n^4)$.

Step three of our algorithm is computing $\mathbf{mdgm}_M$ directly from $\mathbf{mrk}_M$ via the formula in Definition 5.3. As the size of $\mathbf{mrk}_M$ is at most $O(n^4)$, by the definition it is clear that the time complexity of computing $\mathbf{mdgm}_M$ is also $O(n^4)$.

At the end of our algorithm, $\mathbf{mrk}_M$ and $\mathbf{mdgm}_M$ are output as nested lists. Steps one, two, and three each take $O(n^4)$ time, so putting them together we have:

Theorem 5.4. *The meta-rank and meta-diagram can be computed from a simplex-wise bifiltration over P_n in $O(n^4)$ time.*

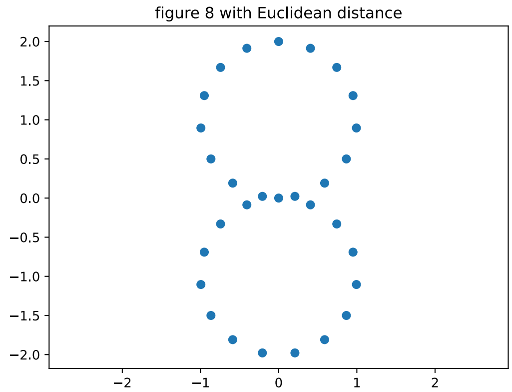


Fig. 6 The figure 8 with 31 points in $\mathbb{R}^2$.

5.2 Visualization Example

Let (X, d_X) be a discretized figure eight seen in Figure 6, with d_X the Euclidean metric inherited from $\mathbb{R}^2$.

Let $f : X \to \mathbb{R}$ be the height function, i.e. $f(x, y) = y$. We compute the meta-rank and meta-diagram of the homology (in all degrees) of the Vietoris-Rips function bifiltration with respect to X and f.

We select the first point (interval) in the outer diagram at which the evaluated meta-diagram is nonzero for one-dimensional homology, with the resulting meta-diagram visualized in Figure 7. This point in the outer diagram is at roughly $(\approx -0.29, 2)$, indicating that at $x \approx -0.29$, enough of the bottom circle in the figure 8 has entered into the filtration to allow the one-dimensional homology class to appear in Vietoris-Rips persistence. The fact that the right endpoint of this interval is 2 indicates that this one-dimensional persistence feature born when the height parameter reaches ≈ -0.29 persists for the remainder of the bifiltration in the x direction. On the right of Figure 7, we see the simplicial complex of the bifiltration when the one-dimensional homology class first appears.

If we evaluate $\mathbf{mdgm}_M$ at the next point to the right in the outer diagram, the resulting inner diagram is visualized in Figure 8. We see the positively signed point in one-dimensional homology has the same death coordinate, but is born earlier. This is because as more of the circle is added, the one-simplex along the top of the circle which completes the loop arrives at an earlier Vietoris-Rips threshold. This can be seen in the simplicial complex of the bifiltration when the one-dimensional homology class appears, plotted on the right side of Figure 8. Again, the fact that the right endpoint of this interval in the outer diagram is 2 indicates that this persistence class persists for the remainder of the filtration in the x direction. The negative point in blue indicates that this one-dimensional feature corresponds to the same feature we saw previously at height ≈ -0.29, but that the persistence of this feature has increased as more points from X were allowed in the filtration.

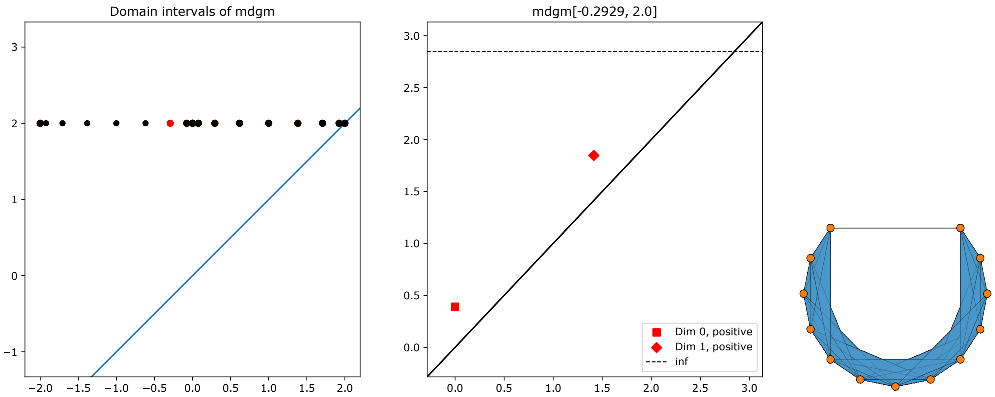


Fig. 7 The first-dimensional homology class of the bottom circle first appears at height ≈-.29.

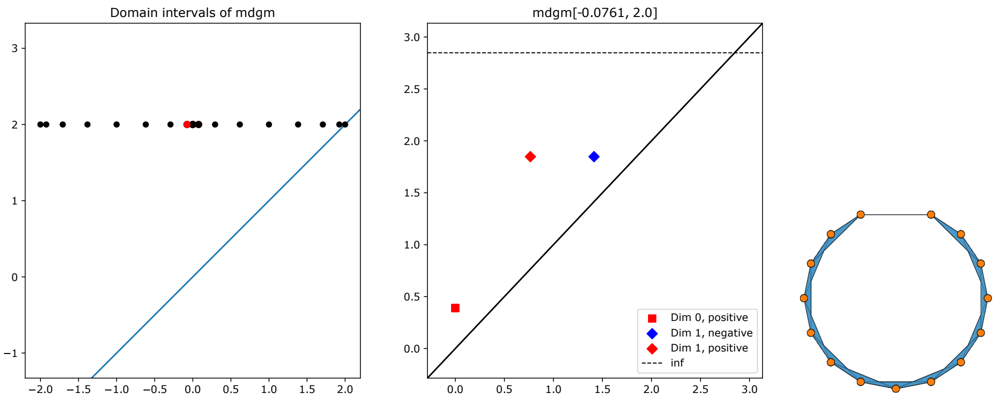


Fig. 8 As more of the bottom circle enters into the bifiltration, the persistence increases.

If we continue and evaluate $\mathbf{mdgm}_M$ at the next point to the right in the outer diagram at $(0, 2)$, the inner diagram updates, with the result seen in Figure 9. Now that all the points in the bottom circle have arrived in the bifiltration, we see that persistence of the one-dimensional homological feature corresponding to this circle is maximized. On the right, we see the simplicial complex of the bifiltration when the one-dimensional homology class appears. Again, the negative point in blue represents that the positively signed point corresponds to the same one-dimensional homology feature we saw previously, but that as more points were added into the bifiltration the persistence of this feature grew.

In Figure 10, Figure 11, and Figure 12, we see a similar phenomenon with tracking the one-dimensional persistence feature corresponding to the top circle of the figure 8 as we allow more points into the bifiltration with the increasing height function parameter. In each figure, on the right hand side is a visualization of the simplicial

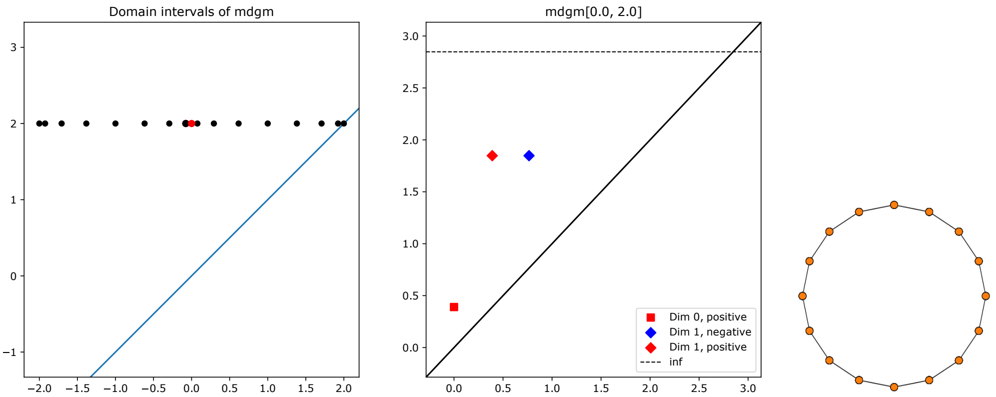


Fig. 9 As the entire bottom circle enters into the bifiltration, the one-dimensional homological feature corresponding to the bottom circle achieves its maximum persistence.

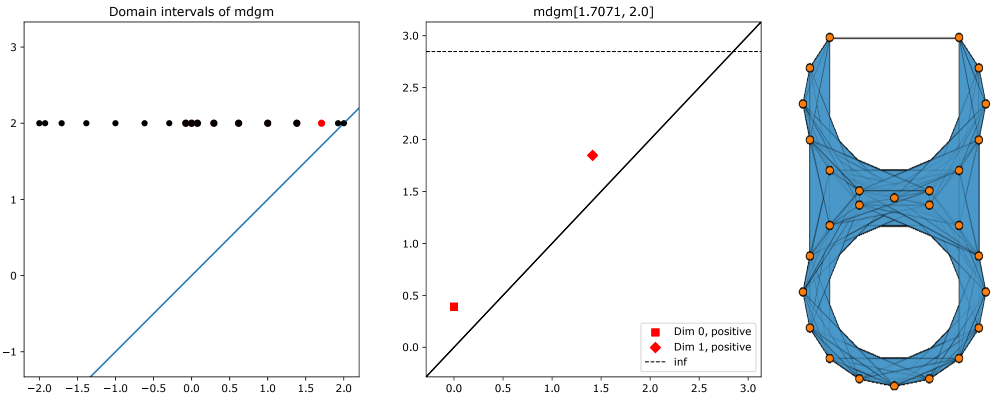


Fig. 10 The first-dimensional homology class of the top circle first appears at height ≈-.29.

complex when the one-dimensional persistence feature first arrives in the bifiltration at the height value corresponding to the lower endpoint of the outer diagram point.

6 Discussion

We conclude with some open questions and threads of future research. First, we would like to extend our approach to the d-parameter setting. We expect that a proper extension would satisfy relationships with the rank invariant and rank decompositions similar to Proposition 3.8 and Proposition 4.11. Such an extension would also lead to a "recursive" formulation of the persistence diagram of diagrams illustrated in Figure 2.

The algorithm we introduce may be limited in applications due to its $O(n^4)$ runtime. A further line of research is looking to develop other algorithms. One particular approach of interest is to use recent work on computing image persistence by Bauer

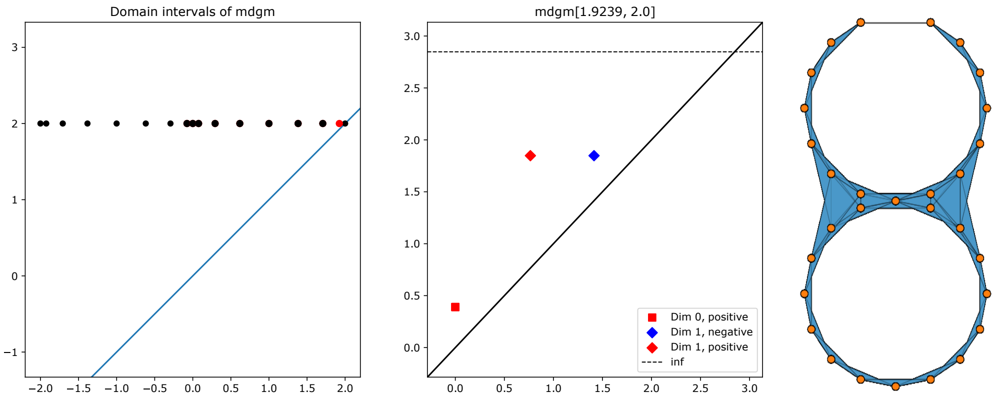


Fig. 11 As more of the top circle enters into the bifiltration, the persistence increases.

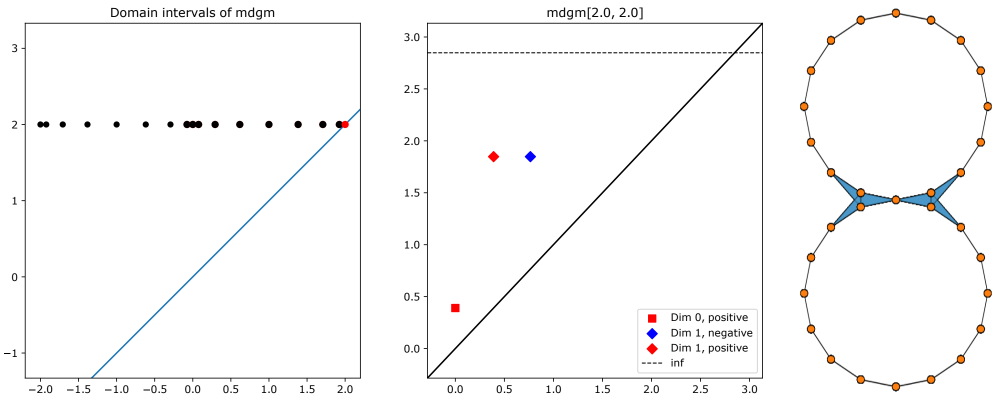


Fig. 12 As the entire figure 8 enters into the bifiltration, the one-dimensional homological feature corresponding to the top circle achieves its maximum persistence.

and Schmahl [15] to compute $\mathbf{mrk}_M(I)$ independently for all $I \in \mathbf{Int}([n])$ independently. While this approach would almost certainly be slower running on one serial process, it is highly parallelizable, and may be faster in practice than the algorithm we introduce in Section 5.

Lastly, there have been multiple recent works that use algorithmic ideas from 1-parameter persistence to compute invariants in the multiparameter setting [9, 14, 25]. We wish to explore in what ways these approaches can create new algorithms or improve upon existing ones for computing the invariants of multi-parameter persistence modules.

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Appendix A Detailed Proofs on Stability Results

A.1 Details for Meta-Rank Stability

Proof of Proposition 3.12. Symmetry is clear from the definition of $\mathrm{d_E}$. It remains to check the triangle inequality. Suppose $M, N, L : \mathbb{R}^2 \to \mathbf{vec}$ are such that $\forall I \in \mathrm{Dgm}$, $\mathbf{mrk}_M(I^{\epsilon_1}_{-\epsilon_1})^{\epsilon_1} \preceq_{2\epsilon_1} \mathbf{mrk}_N(I)$ and $\mathbf{mrk}_N(I^{\epsilon_1}_{-\epsilon_1})^{\epsilon_1} \preceq_{2\epsilon_1} \mathbf{mrk}_M(I)$. Also, suppose $\forall I \in \mathrm{Dgm}$, $\mathbf{mrk}_N(I^{\epsilon_2}_{-\epsilon_2})^{\epsilon_2} \preceq_{2\epsilon_2} \mathbf{mrk}_L(I)$ and $\mathbf{mrk}_L(I^{\epsilon_2}_{-\epsilon_2})^{\epsilon_2} \preceq_{2\epsilon_2} \mathbf{mrk}_N(I)$.

Fix any $I \in \mathrm{Dgm}$. It is clear that $(I^{\epsilon_1}_{-\epsilon_1})^{\epsilon_2}_{-\epsilon_2} = I^{\epsilon_1+\epsilon_2}_{-\epsilon_1-\epsilon_2}$, and so we have:

$$\mathbf{mrk}_M(I^{\epsilon_1+\epsilon_2}_{-\epsilon_1-\epsilon_2})^{\epsilon_1+\epsilon_2} \preceq_{2(\epsilon_1+\epsilon_2)} \preceq \mathbf{mrk}_N(I^{\epsilon_2}_{-\epsilon_2})^{\epsilon_2} \preceq_{2\epsilon_2} \mathbf{mrk}_L(I)$$

and similarly with the roles of M and L reversed. Hence, $\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_L) \leq \epsilon_1 + \epsilon_2$, as desired. □

The following Lemma is useful in the proof of Theorem 3.13:

Lemma A.1. *Let $M : \mathbb{R} \to \mathbf{vec}$ be a persistence module, with barcode* $\mathrm{barc}(M)$*, and let $\epsilon > 0$. Define $M[\epsilon :] : \mathbb{R} \to \mathbf{vec}$ as follows: for $a \leq b \in \mathbb{R}$,*

$$M[\epsilon :](a) := \{x \in M(a) \,|\, \exists x' \in M(a-\epsilon) \; s.t. \; \varphi_M(a-\epsilon \leq a)(x')\}$$

$$M[\epsilon :](a \leq b) := M(a \leq b)|_{M[\epsilon:](a \leq b)}$$

Then $M[\epsilon :] : \mathbb{R} \to \mathbf{vec}$ is a well-defined persistence module, and $\mathrm{barc}_\epsilon(M) = \mathrm{barc}(M[\epsilon :])$*.*

Proof of Lemma A.1. Let $M = \oplus_{I \in \mathcal{I}} \mathbf{k}^I$, and $\{e^t_I\}^{t \in \mathbb{R}}_{I \in \mathcal{I}}$ be such that $\{e^t_I\}_{I \in \mathcal{I}}$ is a basis for $M(t)$ for all t. Further, require $e^t_i \neq 0 \iff t \in I$, and $\varphi_M(s \leq t)(e^s_I) = e^t_I$. The intuition is that each element $e^t_I \in M(t)$ is either 0 or a basis for the summand $\mathbf{k}^I(t)$ of $M(t)$. We call such a set $\{e^t_I\}^{t \in \mathbb{R}}_{I \in \mathcal{I}}$ a *persistence basis*. From the definition, $e^t_I \neq 0 \in M[\epsilon :](t)$ if and only if $e^t_I \neq 0 \in M(t)$ and $e^{t-\epsilon}_I \neq 0 \in M(t-\epsilon)$. Thus, if $\{e^t_{I_1}, \ldots, e^t_{I_n}\}$ is a basis for $M(t)$, then a subset of these will be a basis for $M[\epsilon :](t)$.

For $a \leq b \in \mathbb{R}$, to see that $\varphi_M(a \leq b)|_{M[\epsilon:](a)}$ maps $M[\epsilon :](a)$ into $M[\epsilon :](b)$, we can consider the mapping on basis elements. If $e^a_I \neq 0 \in M[\epsilon :](a)$, then $e^{a-\epsilon}_I \neq 0 \in M(a-\epsilon)$. It follows that

$$\varphi_M(a \leq b)(e^a_I) = \varphi_M(b-\epsilon \leq b)(\varphi_M(a-\epsilon \leq b-\epsilon)(e^{a-\epsilon}_I))$$

and so $\varphi_M(a \leq b)|_{M[\epsilon:](a)}(M[\epsilon :](a) \subseteq M[\epsilon :](b)$, and so $M[\epsilon :]$ is a well-defined persistence module.

Now we show that $\text{barc}_\epsilon(M) = \text{barc}(M[\epsilon :])$. Suppose $I = [s,t) \in \text{barc}_\epsilon(M)$. Then I corresponds uniquely to an interval $[s-\epsilon, t) \in \text{barc}(M)$. Suppose $M = \oplus_{I\in\mathcal{I}} k^I$. This interval in $\text{barc}(M)$ corresponds uniquely to a specific sequence, namely for a fixed $I \in \mathcal{I}$, a sequence of nonzero elements $\{e_I^a\}_{a\in[s-\epsilon,t)}$, with $e_I^a \in M(a) = 0$ for $a \notin [s-\epsilon, t)$. It is straightforward to check that if $\{e_I^t\}_{i\in\mathcal{I}}^{t\in\mathbb{R}}$ is a persistence basis for M, then $\{e_I'^t\}_{I\in\mathcal{I}}^{t\in\mathbb{R}}$ is a persistence vector basis for $M[\epsilon :]$, where $e_I'^t = e_I^t \in M[\epsilon :](t) \iff e_I^t \neq 0 \in M(t)$ and $e_I^{t-\epsilon} \neq 0 \in M(t-\epsilon)$. Otherwise, $e_I'^t := 0$. This means $e_I'^a \neq 0 \in M(a)$ if and only if $e_I^a \neq 0 \in M(a)$ and $e_I^{a-\epsilon} \neq 0 \in M(a-\epsilon) \iff a \in I$. Thus, this sequence $\{e_I^a\}_{a\in[s,t)}$ corresponds uniquely to an interval $[s,t) \in \text{barc}(M[\epsilon :])$. So we have every interval $[s,t) \in \text{barc}_\epsilon(M)$ corresponds uniquely to an interval in $\text{barc}(M[\epsilon :])$, so $\text{barc}_\epsilon(M) \subseteq \text{barc}(M[\epsilon :])$.

To see the reverse containment, if $I = [s,t)$ is an interval in $\text{barc}(M[\epsilon :])$, then we can reverse the previous argument to see that this corresponds uniquely to a sequence of nonzero elements $\{e_I^a\}_{a\in[s-\epsilon,t)}$ in M. This corresponds uniquely to an interval $[s-\epsilon, t)$ in $\text{barc}(M)$, which corresponds to an interval $[s,t) \in \text{barc}_\epsilon(M)$. Hence, $\text{barc}(M[\epsilon :]) \subseteq \text{barc}_\epsilon(M)$, and so $\text{barc}(M[\epsilon :]) = \text{barc}_\epsilon(M)$, as desired. □

Proof of Theorem 3.13. Suppose $\epsilon \geq 0$ and $f : M \to N^{\bar{\epsilon}}$ and $g : N \to M^{\bar{\epsilon}}$ are an interleaving pair with $\bar{\epsilon} = (\epsilon, \epsilon)$. Fix S so that M and N are both S-constructible. Let $I = [s,t) \in \text{Dgm}$. Assume initially that $t \notin S$ and $t+\epsilon \notin S$, these cases will be dealt with at the end.

By the definition of constructibility, we can replace $[s,\infty)$ with $[s,c)$ for some $c \geq s_n$ (recall s_n is the maximal element in S), so we will show the result under the assumption $[s,t) \in \text{Dgm}$, with $t < \infty$, $t \notin S$, and $t + \epsilon \notin S$.

Under our assumption, $\mathbf{mrk}_M(I) = [\text{im}(\phi_x^M(s \leq t))]$, and $\mathbf{mrk}_N(I_{-\epsilon}^{\epsilon}) = [\text{im}(\phi_x^N(s-\epsilon \leq t+\epsilon))]$. Denote by f' the restriction of f to $\text{im}(\phi_x^M(s \leq t))$. Note that f' maps into $N_x^{t+\epsilon}$. We claim that $\text{im}(\phi_x^N(s-\epsilon \leq t+\epsilon))[2\epsilon :] \subseteq \text{im}(f')$.

To see this, let $a \in \mathbb{R}$, and let $x \in \text{im}(\phi_x^N(s-\epsilon \leq t+\epsilon))^\epsilon[2\epsilon :](a)$. By definition, this means there exists $x' \in \text{im}(\phi_x^N(s-\epsilon \leq t+\epsilon))(a-\epsilon)$ such that $\varphi_N((t+\epsilon, a-\epsilon) \leq (t+\epsilon, a+\epsilon))(x') = x$. Further, there is an $x'' \in N_x^{s-\epsilon}(a-\epsilon)$ such that $\varphi_N((s-\epsilon, a-\epsilon) \leq (t+\epsilon, a-\epsilon))(x'') = x'$. Set $y := \varphi_M((s,a) \leq (t,a))(g(x''))$. From this definition, it is clear that $y \in \text{im}(\phi_x^M(s \leq t))$. By the interleaving condition between f and g, we have:

$$\begin{aligned}
f'(y) &= f'(\varphi_M((s,a) \leq (t,a))(g(x'')) \\
&= f'(g(\varphi_N((s-\epsilon, a-\epsilon) \leq (t-\epsilon, a-\epsilon))(x'') \\
&= \varphi_N((t-\epsilon, a-\epsilon) \leq (t+\epsilon, a+\epsilon))(\varphi_N((s-\epsilon, a-\epsilon) \leq (t-\epsilon, a-\epsilon))(x'')) \\
&= \varphi_N((t+\epsilon, a-\epsilon) \leq (t+\epsilon, a+\epsilon))(\varphi_N((s-\epsilon, a-\epsilon) \leq (t+\epsilon, a-\epsilon))(x'')) \\
&= \varphi_N((t+\epsilon, a-\epsilon) \leq (t+\epsilon, a+\epsilon))(x') = x
\end{aligned}$$

As a result, we have a surjective map $f' : \text{im}(\phi_x^M(s \leq t)) \to \text{im}(f')$, and an injective inclusion of persistence modules $\iota : \text{im}(\phi_x^N(s-\epsilon \leq t+\epsilon))[2\epsilon :] \hookrightarrow \text{im}(f')$. By [26] these maps induce injective maps on barcodes $\chi_{f'} : \text{barc}(\text{im}(f')) \hookrightarrow \text{barc}(\mathbf{mrk}_M([s,t)))$ and $\chi_{\iota'} : \text{barc}(\mathbf{mrk}_N([s-\epsilon, t+\epsilon)))^\epsilon[2\epsilon :]) \hookrightarrow \text{barc}(\text{im}(f'))$. By Lemma A.1, we can view χ_ι as a map with domain $\text{barc}_{2\epsilon}(\mathbf{mrk}_N([s-\epsilon, t+\epsilon))^\epsilon)$.

Define $\chi := \chi_{f'} \circ \chi_\iota : \mathrm{barc}_{2\epsilon}(\mathbf{mrk}_N([s-\epsilon, t+\epsilon))^\epsilon) \to \mathbf{mrk}_M([s,t))$. This is injective as it is a composition of injections. For all $J \in \mathrm{barc}_{2\epsilon}(\mathbf{mrk}_N([s-\epsilon, t+\epsilon))^\epsilon)$, we have $\chi(J) = \chi_{f'}(\chi_\iota(J)) \subseteq \chi_\iota(J) \subseteq J$. Thus, $\mathbf{mrk}_N(I^\epsilon_{-\epsilon}) \preceq_{2\epsilon} \mathbf{mrk}_M(I)$. The argument is symmetric when swapping M and N, so we are done with this case.

If $t \in S$, then we can replace t in all the above arguments with $t - \delta$ for some δ small enough such that $t - \delta + \epsilon \notin S$, and the above arguments follow to show $\mathbf{mrk}_N(I^\epsilon_{-\epsilon}) \preceq_{2\epsilon} \mathbf{mrk}_M(I)$.

Lastly, if $t + \epsilon \in S$, then $\mathrm{im}(\phi^N_x(s - \epsilon \leq t + \epsilon)) = \mathbf{mrk}_N([s - \epsilon, t + \epsilon'))$ for all $\epsilon' = \epsilon + \delta$, $\delta > 0$ sufficiently small. Thus, the above arguments give us $\mathbf{mrk}_N(I^{\epsilon'}_{-\epsilon'}) \preceq_{2\epsilon'} \mathbf{mrk}_M(I)$ for all such ϵ', and when taking the infimum in Definition 3.11, we get $\mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N) \leq \epsilon$, as desired. ☐

A.2 Details Meta-Diagram Stability

Proof of Theorem 4.15. To show this, we show $\mathrm{d}^S_{\mathrm{E}}(\mathbf{mdgm}_M, \mathbf{mdgm}_N) \leq \mathrm{d_E}(\mathbf{mrk}_M, \mathbf{mrk}_N)$ and then invoke Theorem 3.13. Let $\epsilon \geq 0$ and suppose that for all $[s,t) \in \mathrm{Dgm}$, we have $\mathbf{mrk}_M([s-\epsilon, t+\epsilon))^\epsilon \preceq_{2\epsilon} \mathbf{mrk}_N([s,t))$, and $\mathbf{mrk}_N([s-\epsilon, t+\epsilon))^\epsilon \preceq_{2\epsilon} \mathbf{mrk}_M([s,t))$. Fix $[s_i, s_j) \in \mathrm{Dgm}$. By our assumption, we have the following four injective maps:

$$\begin{aligned}
\chi_1 &: \mathrm{barc}(\mathbf{mrk}_M([s_i - \epsilon, s_j + \epsilon)))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_N([s_i, s_j)))\\
\chi_2 &: \mathrm{barc}(\mathbf{mrk}_M([s_{i-1} - \epsilon, s_{j+1} + \epsilon)))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_N([s_{i-1}, s_{j+1})))\\
\chi_3 &: \mathrm{barc}(\mathbf{mrk}_N([s_{i-1} - \epsilon, s_j + \epsilon)))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_M([s_{i-1}, s_j)))\\
\chi_4 &: \mathrm{barc}(\mathbf{mrk}_N([s_i - \epsilon, s_{j+1} + \epsilon)))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_M([s_i, s_{j+1})))
\end{aligned}$$

Let $s_a := \overline{S}_{\leq}(s_i - \epsilon)$ and $s_b := \overline{S}_{\geq}(s_j + \epsilon)$. Suppose $c := s_{i+1} - s_i$ (which by assumption is constant for any $1 \leq i \leq n-1$). We then have $s_{a-1} = s_a - c \leq s_i - \epsilon - c = s_{i-1} - \epsilon$. Similarly, we have $s_{b+1} \geq s_{j+1} + \epsilon$. This implies, for example, that $\mathbf{mrk}_M([s_{a-1}, s_{b+1})) \preceq \mathbf{mrk}_M([s_{i-1} - \epsilon, s_{j+1} + \epsilon))$, and a similar statement holds for the domains of the other three maps χ'_i above. Thus, by composing each maps χ_i above with the map guaranteed by the definition of $\preceq$, we can define:

$$\begin{aligned}
\chi'_1 &: \mathrm{barc}(\mathbf{mrk}_M([s_a, s_b)))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_N([s_i, s_j)))\\
\chi'_2 &: \mathrm{barc}(\mathbf{mrk}_M([s_{a-1}, s_{b+1})))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_N([s_{i-1}, s_{j+1})))\\
\chi'_3 &: \mathrm{barc}(\mathbf{mrk}_N([s_{a-1}, s_b)))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_M([s_{i-1}, s_j)))\\
\chi'_4 &: \mathrm{barc}(\mathbf{mrk}_N([s_a, s_{b+1})))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathbf{mrk}_M([s_i, s_{j+1})))
\end{aligned}$$

The multiset union of the four barcodes in the domains form the barcode of $\mathrm{PN}(M,N)([\overline{S}_{\leq}(s_i - \epsilon), \overline{S}_{\geq}(s_j + \epsilon))^\epsilon[2\epsilon :] = \mathrm{PN}(M,N)([s_a, s_b)))^\epsilon[2\epsilon :]$, and the multiset union of the four barcodes in the codomains form the barcode of $\mathrm{PN}(N,M)([s_i, s_j))$. Hence, we can let $\chi : \mathrm{barc}(\mathrm{PN}(M,N))([s_a, s_b))^\epsilon[2\epsilon :] \to \mathrm{barc}(\mathrm{PN}(N,M))([s_i, s_j))$ be the disjoint union of χ'_i for $1 \leq i \leq 4$. As each χ'_i is injective and has $\chi'_i(J) \subseteq J$, these properties will hold for χ as well. ☐

Remark A.2. We can remove the condition S is evenly-spaced, but there is a price to pay for doing so. If S is not evenly-spaced, let $\text{irreg}(S) := (\max_{1 \leq i \leq n-1} s_{i+1} - s_i) - (\min_{1 \leq i \leq n-1} s_{i+1} - s_i)$. We can define erosion distance as before, removing only the evenly-spaced condition. In this setting, the stability result appears as:
Theorem A.3. *Suppose* $M, N : \mathbb{R}^2 \to \mathbf{vec}$ *are* S*-constructible. Then we have:*

$$\mathrm{d}_{\mathrm{E}}^{S}(\mathbf{mdgm}_M, \mathbf{mdgm}_N) \leq \mathrm{d}_{\mathrm{I}}(M, N) + \text{irreg}(S)$$

Note that $\text{irreg}(S) = 0$ if and only if S is evenly-spaced, so this result generalizes Theorem 4.15. The main issue when S is not evenly-spaced is that we could have $s_{a-1} > s_{i-1} - \epsilon$, which causes the proof of Theorem 4.15 to fail. However, the additive term $\text{irreg}(S)$ accounts for this. In particular, set $s_a := \overline{S}_{\leq}(s_i - \epsilon - \text{irreg}(S))$, $c_a := s_a - s_{a-1}$ and $c_i := s_i - s_{i-1}$. By definition, $c_i - c_a \leq \text{irreg}(S)$, so we have:

$$s_{a-1} = s_a - c_a \leq s_i - \epsilon - \text{irreg}(S) - c_a \leq s_i - \epsilon - c_i = s_{i-1} - \epsilon$$

Similarly, setting $s_b := \overline{S}_{\geq}(s_j + \epsilon + \text{irreg}(S))$ we get $s_{b+1} \geq s_{j+1} + \epsilon$. The proof of Theorem A.3 then follows similarly to that of Theorem 4.15, upon using our new definitions for s_a and s_b.

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