Robustness for 2D Symmetric Tensor Field Topology

Bei Wang and Ingrid Hotz

Abstract Topological feature analysis is a powerful instrument to understand the essential structure of a dataset. For such an instrument to be useful in applications, however, it is important to provide some importance measure for the extracted features that copes with the high feature density and discriminates spurious from important structures. Although such measures have been developed for scalar and vector fields, similar concepts are scarce, if not nonexistent, for tensor fields. In particular, the notion of robustness has been proven to successfully quantify the stability of topological features in scalar and vector fields. Intuitively, robustness measures the minimum amount of perturbation to the field that is necessary to cancel its critical points.

This chapter provides a mathematical foundation for the construction of a feature hierarchy for 2D symmetric tensor field topology by extending the concept of robustness, which paves new ways for feature tracking and feature simplification of tensor field data. One essential ingredient is the choice of an appropriate metric to measure the perturbation of tensor fields. Such a metric must be well-aligned with the concept of robustness while still providing some meaningful physical interpretation. A second important ingredient is the index of a degenerate point of tensor fields, which is revisited and reformulated rigorously in the language of degree theory.

1 Introduction

As a linear approximation of physical phenomena, tensors play an important role in numerous engineering, physics, and medical applications. Examples include various

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descriptors of stress at a point in a continuous medium under load or the diffusion characteristics of water molecules in fibrous media. Tensors provide a powerful and simple language to describe anisotropic phenomena for which scalars and vectors are not sufficient, but the analysis of tensor fields is a complex and challenging task. Therefore, visualization becomes a crucial capability to support the understanding of tensor fields. See [18] for a survey on the analysis and visualization of second-order tensors.

In this chapter, we are especially interested in a structural characterization of symmetric second-order tensor fields using topological methods, which can form the basis of advanced analysis and visualization methods. Roughly speaking, tensor field topology segments the tensor field into regions of equivalent tensor line behavior. Conceptually, it is closely related to the vector field topology. Degenerate points in tensor fields take the role of critical points in vector fields, and tensor lines correspond to streamlines. However, despite these parallels, there are also many differences.

First, whereas critical points in vector fields behave as sources and sinks and separatrices can be interpreted as material boundaries of flows, the topological features of tensor fields often do not have a direct physical meaning. Degenerate points are points of high symmetry with isotropic behavior and thus might be considered as being especially boring. However, they play an important role from a structural point of view, as they are points where the eigenvector field is not uniquely specified and thus not necessarily continuous.

Second, there are also major structural differences in comparison to vector fields, because eigenvector fields have no specified orientation. In the 2D case, they exhibit a rotational symmetry with a rotational angle of π. As such, they are a special case of N-symmetric direction fields [17], which are important for many applications in geometry processing and texture design. For example, the eigenvector fields of the curvature tensor have been used for the purpose of quadrangular re-meshing, where degenerate points are mesh vertices with distinct valency [1, 16]. A related application is the synthesis of textures, for example, by defining the stroke directions as an eigenvector field of some tensor field, where degenerated points account for points with non-trivial texture characteristics [30, 3]. For both applications, it is essential to have control over the number of degenerate points. Furthermore, in tensor field analysis, it can also be beneficial to have control over not only the degenerated points but also their cancellation for feature-preserving interpolation and smoothing [15, 24].

While tensor field topology has attracted the most attention in geometric applications, it was introduced along with the vector field topology in visualization applications by Delmarcelle [9] and Tricoche [26]. Since the introduction of tensor field topology, theoretical and application-driven advancement has been slow for several reasons (in contrast to vector field topology): the lack of theory for 3D tensor fields, the complexity of the resulting topological structures, and the challenge of a direct interpretation of such structures in the application domain. However, there has been some recent, encouraging effort by Zhang et al. [31] concerning a theory for 3D tensor field and the application of stress tensor field analysis. To further develop tensor field topology as a useful analysis tool, we are convinced that a major requirement is to find a mathematically rigorous way to cope with the high density of the extracted topological features, even in the setting of 2D tensor fields, where a large number of structures originate from extended isotropic regions and are very sensitive to small changes in the data.

The most important requirement in applications is a *stable* topological skeleton representing the core structure of the data. For all the above-mentioned applications, tensor field topology can provide a means for the controlled manipulation and simplification of data. In this work, we introduce a measure for the stability of degenerate points with respect to small perturbations of the field. The measure is based on the notion of robustness and well group theory, which has already been successfully applied to vector fields. We extend this concept to 2D symmetric second-order tensor fields to lay the foundation for a discriminative analysis of essential and spurious features.

The work presented in this chapter paves the way for a complete framework of tensor field simplification based on robustness. We generalize the theory of robustness to the space of analytical tensor fields. In particular, we discuss the appropriate metrics for measuring the perturbations of tensor fields, but a few challenges remain. First, we need to develop efficient and stable algorithms to generate a hierarchical scheme among degenerate points. Second, the actual simplification of the tensor field using the hierarchical representation and cancellation of degenerate points is technically non-trivial. In this chapter, we focus on the first part by providing the necessary foundation for the following steps.

Our main contributions are threefold: First, we interpret the notion of tensor index under the setting of degree theory; Second, we define tensor field perturbations and make precise connections between such perturbations with the perturbations of bidirectional vector fields; Third, we generalize the notion of robustness to the study of tensor field topology.

This chapter is structured as follows: After reviewing relevant work in Section 2, we provide a brief description of well group theory and robustness for vector fields in Section 3. Then in Section 4, we reformulate some technical background in tensor field topology in a way that is compatible with robustness, by introducing the bidirectional vector field and an anisotropy vector field. The anisotropy vector field then provides the basis for Section 5 in which the notion of robustness is extended to the tensor field setting.

2 Related work

Tensor Field Topology. Previous research has examined the extraction, simplification, and visualization of the topology of symmetric second order tensor fields on which this work builds. The introduction of topological methods to the structural analysis of tensor fields goes back to Delmarcelle [9]. In correspondence to vector field topology, Delmarcelle has defined a topological skeleton, consisting of degenerate points and separatrices, which are tensor lines connecting the degenerate points. Delmarcelle has mainly been concerned with the characterization of degenerate points in two-dimensional fields. Therefore he also provided a definition for the index of a critical point. Tricoche et al. [27] built on these ideas by developing algorithms to apply the concept of topological skeleton to real data. A central question of their work is the simplification of the tensor field topology and tracking it over time. They succeeded in simplifying the field, but the algorithm contains many parameters and is very complex. A robust extraction and classification algorithm for degenerate points has been presented by Hotz et al. [15]. Their method is based on edge labeling using an eigenvector-based interpolation. This work has been extended by Auer et al. [2] to cope with the challenge of discontinuities of tensor fields on triangulated surfaces. While the characteristics of the tensor field topology for two-dimensional fields are similar to the vector field topology, it is in general not possible to define a global vector field with the same topological structure. It is possible, however, to define a vector field whose critical points are located in the same positions as the degenerate points of the tensor field by duplicating their indexes. This idea has been used by Zhang et al. [30] for constructing a simplified tensor field for texture generation. Our method follows a similar line of thought but goes a step further by defining an isometric mapping of the tensor field to a vector field.

Robustness for Vector Fields. In terms of vector field topology, topological methods have been employed extensively to extract features such as critical points and separatrices for vector field visualization [19] and simplification [8]. Motivated by hierarchical simplification of vector fields, the topological notion of *robustness* has been used to rank the critical points by measures of their stability. Robustness is closely related to the notion of *persistence* [10]. Introduced via the algebraic concept of well diagram and well group theory [11, 12, 7], it quantifies the stability of critical points with respect to the minimum amount of perturbation in the fields required to remove them. Robustness has been shown to be very useful for the analysis and visualization of 2D and 3D vector fields [28, 23]. In particular, it is the core concept behind simplifying a 2D vector field with a hierarchical scheme that is independent of the topological skeleton [22]; and it leads to the first ever 3D vector field simplification, based on critical point cancellation [20]. Measures of robustness also lead to a fresh interpretation of critical point tracking [21]: Stable critical points can be tracked more easily and more accurately in the time-varying setting.

In this paper, we extend the notion of robustness to the study of tensor field topology. We would like to rely on such a notion to develop novel, scalable, and mathematically rigorous ways to understand tensor field data, especially questions pertaining to their structural stability. We believe that robustness holds the key to increase the interpretability of tensor field data, and may lead to a new line of research that spans feature extraction, feature tracking, and feature simplification of tensor fields.

3 Preliminaries on Robustness for Vector Fields

In this section, we briefly review the relevant technical background of robustness for 2D vector field such as critical points, degrees, indices, well groups and well diagrams. These concepts are important for developing and understanding the extensions of robustness for the tensor field.

Critical Point and Sublevel Set. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous vector field. A *critical point* of f is a zero of the field, i.e., $f(x) = 0$. Define $f_0 : \mathbb{R}^2 \to \mathbb{R}$ as the vector magnitude of f, $f_0(x) = ||f(x)||_2$, for all $x \in \mathbb{R}^2$. Let $\mathbb{F}_r$ denote the sublevel set of f_0, $\mathbb{F}_r = f_0^{-1}(-\infty, r]$, that is, all points in the domain with a magnitude up to r. In particular, $\mathbb{F}_0 = f^{-1}(0)$ is the set of critical points. A value $r > 0$ is a *regular value* of f_0 if $\mathbb{F}_r$ is a 2-manifold, and for all sufficiently small $\varepsilon > 0$, $f_0^{-1}[r-\varepsilon, r+\varepsilon]$ retracts to $f_0^{-1}(r)$; otherwise it is a *critical value*. We assume that f_0 has a finite number of critical values and f contains a finite number of isolated critical points. Fig. 1 gives an example of a 2D vector field f with four critical points (Fig. 1 left) and the regions in the domain enclosed by colored contour lines of f_0 (Fig. 1 middle) illustrate sublevel sets of f_0 at critical values.

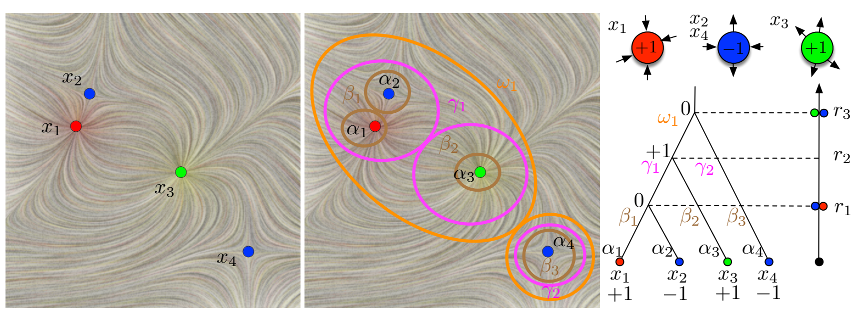

Fig. 1: Figure recreated from [28] showing the merge tree for a continuous 2D vector field example. From left to right: vector fields f, relations among components of $\mathbb{F}_r$ (for $r \geq 0$), and the augmented merge tree. f contains four critical points, a red sink x_1, a green source x_3, and two blue saddles x_2 and x_4. We use β, γ, ω, etc., to represent components of the sublevel sets.

Degree and Index. Suppose x is an isolated critical point of f. For a 2D vector field, the *degree* of x equals its index, which in turn corresponds to the winding number of a simple closed curve on the plane around x. Formally, fix the local coordinates near x and pick a closed disk D that encloses x in its interior and contains no other critical points. Then the *index* of x (w.r.t. f), $\mathrm{I}_f(x)$, or equivalently the *(local) degree* of f at x, denoted as $\deg(f\mid_x)$, is the degree of the mapping $u : \partial D \to \mathbb{S}^1$ that associates

∂D (the boundary of D) to the circle, given by $u(z) = f(z)/|f(z)|$ (u is sometimes referred to as the *Gauss* map). It is shown that isolated first-order critical points have an index of ± 1: a saddle has an index of -1 and non-saddles have an index of $+1$. In Fig. 1 right, x_2, x_4 are saddles of index -1 whereas x_1 and x_3 have index $+1$.

Let $C \subseteq \mathbb{F}_r$ be a path-connected component of $\mathbb{F}_r$. Consider $\{x_1, x_2, ..., x_n\}$ to be the set of critical points in C. Then the degree of f restricted to ∂C is the sum of the degrees of f at the x_i, $\deg(f\,|_{\partial C}) = \sum_{i=1}^{n} \deg(f\,|_{x_i})$. For notational convenience, when f is fixed, we abuse the notation by defining the degree of C as $\deg(C) := \deg(f\,|_{\partial C})$. For example, in Fig. 1 middle, component β_1 (representing a sublevel set of f_0) is of degree 0 as it contains critical points x_1 and x_2 with opposite degrees.

Poincaré-Hopf Theorem for Vector Fields. We review the Poincaré-Hopf theorem in the setting of a 2D vector field. A particularly useful corollary for 2D vector field simplification is that if a region $C \subset \mathbb{R}^2$ has degree zero, it is possible to replace the vector field inside C with a vector field free of critical points.

Theorem 1 (Poincaré-Hopf theorem). *Let $\mathbb{M}$ be a smooth compact 2-manifold. Let f be a vector field on $\mathbb{M}$ with finitely many isolated critical points. For $\mathbb{M}$ with boundary, f points in the outward normal direction along the boundary. Then the sum of the indices of the critical points is equal to the Euler characteristic of the manifold:* $\sum_i \mathrm{I}_f(x_i) = \chi(\mathbb{M})$.

Well Group. Given a mapping $f : \mathbb{X} \to \mathbb{Y}$ and a subspace $A \subseteq \mathbb{Y}$, the well group theory [11, 12, 7] studies the robustness of the homology of the pre-image of A, $f^{-1}(A)$ with respect to perturbations of the mapping f. Roughly speaking, the homology of a topological space measures its topological features, where the rank of the 0-, 1- and 2-dimensional homology groups corresponds to the number of connected components, tunnels, and voids, respectively. Here we review the well group theory in the setting of a 2D vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ where $A = 0$, and correspondingly study the stable property of the critical points ($f^{-1}(0)$) of f [7].

Let $f, h : \mathbb{R}^2 \to \mathbb{R}^2$ be two continuous 2D vector fields. Define the distance between the two mappings as $\mathrm{d}(f,h) = \sup_{x \in \mathbb{R}^2} ||f(x) - h(x)||_2$. We say a continuous mapping h is an *r-perturbation* of f, if $\mathrm{d}(f,h) \leq r$. In other words, for each point $x \in \mathbb{R}^2$, the point $h(x)$ lies within a disk of radius r centered at $f(x)$. See Fig. 2.

If h is an r-perturbation of f, then $h^{-1}(0)$ is a subspace of $\mathbb{F}_r$, that is, we have an inclusion $h^{-1}(0) \subseteq \mathbb{F}_r$. The connected components of $h^{-1}(0)$ generate a vector space that is the 0-dimensional homology group of $h^{-1}(0)$, denoted as $\mathsf{H}(h^{-1}(0))$. Similarly, we have the 0-dimensional homology group of $\mathbb{F}_r$, denoted as $\mathsf{H}(\mathbb{F}_r)$. The subspace relation $h^{-1}(0) \subseteq \mathbb{F}_r$ induces a linear map $j_h : \mathsf{H}(h^{-1}(0)) \to \mathsf{H}(\mathbb{F}_r)$ between the two vector spaces. The *well group*, $\mathsf{U}(r)$, as first studied in [12], is the subgroup of $\mathsf{H}(\mathbb{F}_r)$ whose elements belong to the image of each j_h for all r-perturbation h of f. That is,

$$\mathsf{U}(r) = \bigcap_h \mathrm{im}\, j_h. \tag{1}$$

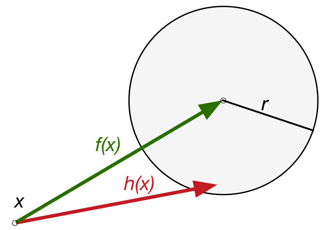

Fig. 2: Geometric interpretation of an r-perturbation of a vector field at a point x in the domain.

Assuming a finite number of critical points, the rank of $\mathsf{U}(0)$ is the number of critical points of f. For values $r < s$, $\mathbb{F}_r \subseteq \mathbb{F}_s$ inducing a linear map $\mathsf{f}_r^s : \mathsf{H}(\mathbb{F}_r) \to \mathsf{H}(\mathbb{F}_s)$ between the two homology groups. It can be shown that $\mathsf{U}(s) \subseteq \mathsf{f}_r^s(\mathsf{U}(r))$, for $r \leq s$. Therefore the rank of the well group decreases monotonically as r increases. The following lemma suggests an algorithm to compute the rank of the well groups.

Lemma 1 (Lemma 3, [7]). *If r is a regular value of f_0, then the rank of the well group $\mathsf{U}(r)$ is the number of connected components $C \subseteq \mathbb{F}_r$ such that $\deg(C) \neq 0$.*

Well Diagram. A point r belongs to the *well diagram* of f_0, $\mathrm{Dgm}(f_0)$, with multiplicity k if the rank of the well group drops by k at r [7]. For reasons of stability, the point 0 is counted with infinite multiplicity. The point ∞ is counted with multiplicity k if for all sufficiently large values of r, the rank of $\mathsf{U}(r)$ is k. The well diagram contains a multi-set of points (infinitely many points at 0 and finite number of nonzero points) on the extended real line, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, where each point in $\mathrm{Dgm}(f_0)$ is either a 0, a positive real number, or ∞.

We therefore consider each point in the well diagram as a measure of how *resistant* a homology class of $f^{-1}(0)$ is against perturbations of the mapping f [12]. Recall that f_0 has finitely many critical values that can be indexed consecutively as $\{r_i\}_i$ (where $0 = r_0 < r_1 < r_2 < \cdots < r_l$), $\mathsf{U}(r_i) \subseteq \mathsf{F}(r_i) := \mathsf{H}(\mathbb{F}_{r_i})$ are the corresponding well groups. We define the mapping $\mathsf{f}_0^j : \mathsf{F}(0) \to \mathsf{F}(r_j)$. A homology class α in the well group $\mathsf{U}(0)$ *dies* at r_j if $\mathsf{f}_0^i(\alpha)$ is a nonzero class in $\mathsf{U}(r_i)$; and either $\mathsf{f}_0^j(\alpha) = 0$, or $\mathsf{f}_0^j(\alpha) \notin \mathsf{U}(r_j)$, for each $i < j$. The *robustness* of a class α in $\mathsf{U}(0)$ is the value at which the class dies [12].

As shown in the example of Fig. 1 right, each critical point of f generates a class in $\mathsf{U}(0)$, denoted as α_1, α_2, α_3, and α_4 (corresponding to critical points x_1 to x_4, respectively). At r_1, two classes α_1 and α_2 die, and therefore they have a robustness of r_1. Similarly, α_3 and α_4 die at r_3, with a corresponding robustness of r_3. In terms of the well diagram, the well group $\mathsf{U}(r_1)$ drops in rank by two because there is an r_1-perturbation of f such that there are only two zeros. Therefore two points are in the well diagram at r_1. Similarly, two points in the well diagram at r_3 because the well group drops its rank by two.

Robustness of Critical Points. In the setting of 2D vector fields, the robustness of a critical point x_i can be described by the robustness of the class α_i in $\mathsf{U}(0)$ that it generates [1]. Therefore in our example (Fig. 1), points x_1, x_2, x_3, and x_4 have robustness r_1, r_1, r_3 and r_3 respectively.

To compute the robustness of critical points in f, we construct an *augmented merge tree* of f_0 that tracks the (connected) components of $\mathbb{F}_r$ together with their degree information as they appear and merge by increasing r from 0. A leaf node represents the creation of a component at a local minima of f_0 and an internal node represents the merging of components. See [28, 6] for algorithmic details. The *robustness* of a critical point is the height of its lowest degree zero ancestor in the merge tree. To illustrate the construction, we show a 2D example recreated from [28] in Fig. 1. By definition, the critical points x_1 and x_2 have robustness r_1, whereas x_3 and x_4 have robustness r_3. Such a topological notion quantifies the stability of a critical point with respect to perturbations of the vector fields. Intuitively, if a critical point x has robustness r, then it can be canceled with a $(r+\delta)$-perturbation, but not with any $(r-\delta)$-perturbation, for $\delta > 0$ arbitrarily small.

Given the above machineries, the properties associated with robustness for critical points are direct consequences of Lemma 2 and Lemma 3. Their original proof sketches can be found in the supplementary material of [28]. These proofs, which are similar to the proof of Lemma 1, are revisited in Section 5.3 for completeness (in the setting of a specific type of vector field).

Lemma 2 (Nonzero Degree Component for Vector Field Perturbation, Corollary 1.2 in [28] supplement). *Let r be a regular value of f_0 and C a connected component of $\mathbb{F}_r$ such that $\deg(C) \neq 0$. Then for any δ-perturbation h of f, where $\delta < r$, the sum of the degrees of the critical points in $h^{-1}(0) \cap C$ is $\deg(C)$.*

Lemma 3 (Zero Degree Component for Vector Field Perturbation, Corollary 1.1 in [28] supplement). *Let r be a regular value of f_0 and C a connected component of $\mathbb{F}_r$ such that $\deg(C) = 0$. Then, there exists an r-perturbation h of f such that h has no critical points in C, $h^{-1}(0) \cap C = \emptyset$. In addition, h equals f except possibly within the interior of C.*

In the example of Fig. 1 right, x_1 has a robustness of r_1, Lemma 3 implies that there exists an $(r_1+\delta)$-perturbation (for an arbitrarily small $\delta > 0$) that can cancel x_1 by locally modifying the connected component $C \subseteq \mathbb{F}_{r_1+\delta}$ containing it.

4 Tensor Fields and Bidirectional Anisotropy Vector Fields

For the remainder of this paper, we consider 2D symmetric second-order tensor fields. In this section, we establish the necessary foundations for introducing a robustness measure for the degenerate points of tensor fields. We introduce the notion

[1] We rely on this definition to describe the robustness of a critical point x_i, even though the critical point is only a particularly chosen generator of the class α_i in $\mathsf{U}(0)$.

of *bidirectional anisotropy vectors*, which serves two purposes. First, we use it to define the notion of perturbations of tensor fields for our setting (Section 4.2). Second, this notion will be central for the definition of the tensor index under the setting of degree theory (Section 5.1).

We start by summarizing some basic concepts of tensor field topology in Section 4.1. For a complete introduction, we refer the reader to the work by Delmarcelle [9] or Trichoche [25]. Then we introduce the notion of bidirectional anisotropy vector fields in Section 4.2 and discuss its relation with respect to the space of deviators. Finally, we establish an isometry from the space of deviators to the anisotropy vector field in Section 4.3.

4.1 Background in Tensor Field Topology

The topology of a 2D symmetric second-order tensor field is defined as the topology of one of the two eigenvector fields [9]. The degenerate points constitute the basic ingredient of the tensor field topology and play a role similar to that for the critical points (zeros) for vector fields.

2D Symmetric Second-Order Tensor Fields. In our setting, a *tensor* T is a linear operator that associates any vector v to another vector $u = Tv$, where v and u are vectors in the Euclidean vector space $\mathbb{R}^2$. In this work, we restrict ourselves to symmetric tensors. A *tensor field* $\mathbf{T}$ assigns to each position $x = (x_1, x_2) \in \mathbb{R}^2$ a symmetric tensor $\mathbf{T}(x) = T$. Let $\mathscr{T}$ denote the space of 2D symmetric second-order tensors over $\mathbb{R}^2$. In matrix form, with respect to a given basis of $\mathbb{R}^2$, a tensor field $\mathbf{T}$ is defined as

$$\mathbf{T} : \mathbb{R}^2 \to \mathscr{T}, \mathbf{T}(x) = T = \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}. \tag{2}$$

The tensor T at x is fully specified by two orthogonal eigenvectors v_i at x and its two associated real eigenvalues λ_i, defined by the eigenvector equation $Tv_i = \lambda_i v_i$ (for $i \in \{1,2\}$) with $v_i \in \mathbb{R}^2$ and $v_i \neq 0$. By imposing an ordering of $\lambda_1 \geq \lambda_2$, the normalized eigenvectors e_1 (resp. e_2) associated with λ_1 (resp. λ_2) are referred to as the *major* (resp. *minor*) eigenvectors.

Degenerate Points. At points x where the eigenvalues of $\mathbf{T}(x)$ are different $\lambda_1 \neq \lambda_2$, the eigenspace of λ_i (for $i \in \{1,2\}$) is the union of the zero vector and the set of all eigenvectors corresponding to eigenvalue λ_i, which is a one-dimensional subspace of $\mathbb{R}^2$. Such points are considered *non-degenerate points* of the tensor field $\mathbf{T}$. At these points, the tensor can then be expressed as

$$T = \lambda_1 e_1 \otimes e_1 + \lambda_2 e_2 \otimes e_2 \tag{3}$$

where $\otimes$ denotes the tensor product of the normalized eigenvectors e_i. For point $x_0 \in \mathbb{R}^2$ where $\lambda_1(x_0) = \lambda_2(x_0) = \lambda$, its associated tensor is proportional to the unit tensor and the corresponding eigenspace is the entire vector space $\mathbb{R}^2$. Its matrix representation is independent from the frame of reference, given as

$$T(x_0) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

The points x_0 are called *degenerate points*. In the following sections, we assume that these points are isolated points in $\mathbb{R}^2$, which is usually the case. While the degenerate points as isotropic points exhibiting a high symmetry, they are structurally the most important features for the eigenvector fields.

Real Projective Line. Before we proceed, we need the notion of real projective line and the homeomorphism between the real projective line and the circle. The real projective line, denoted as $\mathbb{RP}^1$ (or $\mathbb{P}^1$ for short), can be thought of as the set of lines through the origin of $\mathbb{R}^2$, formally $\mathbb{P}^1 := (\mathbb{R}^2 \setminus \{0\})/\sim$, for the equivalence relation $x \sim y$ iff $x = cy$ for some nonzero $c \in \mathbb{R}$. We sketch the proof below for $\mathbb{P}^1$ being homeomorphic to a circle $\mathbb{S}^1$, via $\mathbb{P}^1 \simeq (\mathbb{S}^1/\sim) \simeq \mathbb{S}^1$.

The quotient topology of a real projective line can be described by the mapping $\eta : \mathbb{R}^2 \setminus \{0\} \to \mathbb{P}^1$ that sends a point $x \in \mathbb{R}^2 \setminus \{0\}$ to its equivalent class $[x]$. η is surjective and has the property that $\eta(x) = \eta(y)$ iff $x \sim y$. Restricting such a mapping to $\mathbb{S}^1$, we obtain a mapping $\eta|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{P}^1$ that identifies the two antipodal points. We now consider $\mathbb{S}^1$ as $\{z \in \mathbb{C} \mid ||z|| = 1\}$. Then we have $\eta|_{\mathbb{S}^1}(z) = [z]$. It is easy to show that $\eta|_{\mathbb{S}^1}$ defines a homeomorphism between $\mathbb{P}^1$ and $\mathbb{S}^1/\sim$ (where $\sim$ describes the equivalence of $z \sim -z$) since $\eta|_{\mathbb{S}^1}$ has the property that $U \subset (\mathbb{S}^1/\sim)$ is open (w.r.t. quotient topology on $\mathbb{S}^1/\sim$) iff $(\eta|_{\mathbb{S}^1})^{-1}(U)$ is open in $\mathbb{R}^2 \setminus \{0\}$.

Now consider the mapping $\theta : \mathbb{S}^1 \to \mathbb{S}^1$ defined as $\theta(z) = z^2$. θ is a continuous surjective function such that $\theta(z) = \theta(-z)$. Following the universal property (of quotient topology[2]), there exists a unique continuous homeomorphism $\phi : (\mathbb{S}^1/\sim) \to \mathbb{S}^1$ by having θ descending to the quotient. Therefore $(\mathbb{S}^1/\sim) \simeq \mathbb{S}^1$.

Eigenvector Fields. In the following section, we describe the construction of an eigenvector field associated with the tensor field $\mathbf{T}$. As described before, a real 2D symmetric tensor T at x has two (not necessarily distinct) real eigenvalues $\lambda_1 \geq \lambda_2$ with associated eigenvectors v_1 and v_2. It is important to note that neither norm nor orientation is defined for the eigenvectors via the eigenvector equation, that is, if v_i is an eigenvector, then so is cv_i for any nonzero $c \in \mathbb{R}$. The normalized eigenvectors are denoted as e_i (for $i \in \{1,2\}$), where $e_i \in \mathbb{S}^1$. This point of view is reflected through the interpretation of an eigenvector as elements of the real projective line. Thus we define the two eigenvector fields as the mapping $\boldsymbol{\psi}_i : \mathbb{R}^2 \to \mathbb{P}^1$ (for $i \in \{1,2\}$), referred to as the major and minor eigenvector fields, respectively:

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[2] The quotient space $\mathbb{X}/\sim$ together with the quotient map $q : \mathbb{X} \to (\mathbb{X}/\sim)$ is characterized by the following universal property: If $g : \mathbb{X} \to \mathbb{Z}$ is a continuous map such that $a \sim b$ implies $g(a) = g(b)$ for all a and b in $\mathbb{X}$, then there exists a unique continuous map $f : (\mathbb{X}/\sim) \to \mathbb{Z}$ such that $g = f \circ q$. We say that g descends to the quotient.

$$\boldsymbol{\psi}_i : \mathbb{R}^2 \to \mathbb{P}^1, x \mapsto \begin{cases} [e_i] & \text{if } \lambda_1 \neq \lambda_2 \\ [e_0] & \text{for degenerate points, if } \lambda_1 = \lambda_2 \end{cases} \quad (4)$$

$[e_0]$ is an arbitrary chosen element of $\mathbb{P}^1$. Note that the eigenvector field is not continuous in degenerate points, and in general it is not possible to define $[e_0]$ such that it becomes continuous. From now on, we restrict our attention to the major eigenvector field, referred to as the *eigenvector field* of $\mathbf{T}$, denoted as $\boldsymbol{\psi} := \boldsymbol{\psi}_1$, as the minor eigenvectors are always orthogonal and do not provide additional information in the 2D case.

The eigenvector fields and the degenerate points constitute the basic ingredients of the topological structure of a tensor field, and they build the basics for the bidirectional anisotropy vector fields that will be defined in Section 4.2.

4.2 Space of Bidirectional Anisotropy Vectors

In this section, we define the space of *bidirectional anisotropy vectors* equipped with a distance measure that is based on the L_2 norm of vectors. A comparison with the commonly used distance measure for tensors using the Frobenius norm shows that this space is topological equivalent to the space of deviators $\mathscr{D}$. The bidirectional anisotropy vectors constitute a step toward the definition of an *anisotropy vector field* later used in the study of robustness.

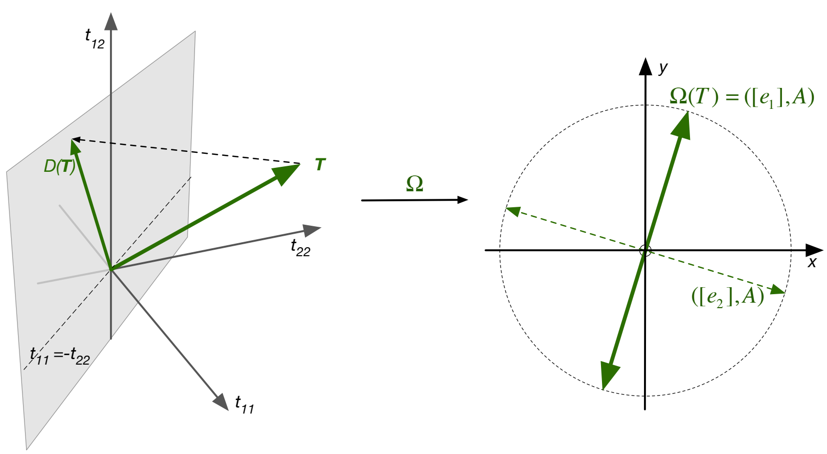

Fig. 3: Mapping of a tensor T to the bidirectional anisotropy vector defined by its anisotropy A and the eigenvector $[e_1]$. The space on the right represents the vector space spanned by the three independent components of the tensor. The gray plane on the left highlights the subspace of traceless tensors.

Bidirectional Anisotropy Vectors. We define bidirectional anisotropy vectors as bidirectional vectors ω whose direction is defined by the equivalence class of the major eigenvector $[e_1]$ and a norm given by the tensors anisotropy A (e.g., $A = |\lambda_1 - \lambda_2|$). Formally, we consider these vectors as elements of $\mathbb{P}^1 \times \mathbb{R}_{\geq 0}$. Degenerate points, that is, points with zero anisotropy and an undefined major eigenvector, are represented as the zero vectors.

Let $\mathscr{T}$ be the space of 2D symmetric tensors over $\mathbb{R}^2$. For each tensor $T \in \mathscr{T}$, we define the bidirectional anisotropy vector by the following mapping (Fig. 3):

$$\Omega : \mathscr{T} \to \mathbb{P}^1 \times \mathbb{R}_{\geq 0}$$
$$T \mapsto \Omega(T) = \omega = \begin{cases} ([e_1], A) \text{ if } \lambda_1 \neq \lambda_2 \\ ([e_0], 0) \text{ for degenerate points, if } \lambda_1 = \lambda_2 \end{cases} \tag{5}$$

The space $\mathbb{P}^1 \times \mathbb{R}_{\geq 0}$ can also be interpreted as $(\mathbb{R}^2/\sim)$, for the equivalence relation $x \sim y$ iff $x = -y$. In this setting, ω is equal to the equivalence class $[Ae_1] = \{v, -v\}$ consisting of the two vectors $v = Ae_1 \in \mathbb{R}^2$ and $-v = -Ae_1 \in \mathbb{R}^2$ with $e_1 \in [e_1]$.

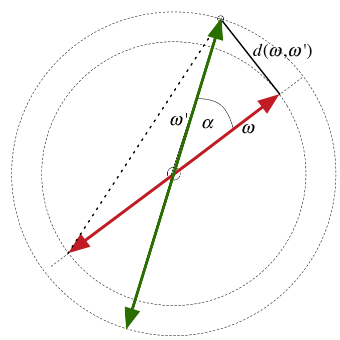

Fig. 4: The distance between two bidirectional vectors ω and ω' defined as the minimal distance between the members of their equivalence classes.

Distance Measure. We now define a distance measure between two bidirectional vectors $\omega = \{v, -v\}$ and $\omega' = \{v', -v'\}$ with vector representatives v and v', respectively. See Fig. 4 for an illustration:

$$d(\omega', \omega) = \min(\|v - v'\|_2, \|v + v'\|_2). \tag{6}$$

Theorem 2. *The distance measure defined in Eq. (6) is a metric on the space of bidirectional vectors.*

Proof. The expression in Eq. (6) is obviously independent on the arbitrarily chosen representatives v and v'. It is also obviously symmetric and non-negative. Therefore we have: $\mathrm{d}(\omega,\omega') = 0 \Leftrightarrow \min(\|v-v'\|_2, \|v+v'\|_2) = 0 \Leftrightarrow v = v'$ or $v = -v' \Leftrightarrow \omega = \omega'$. Furthermore the triangle inequality is satisfied (see Appendix B for derivations). Thus Eq. (6) defines a metric on the space of bidirectional vector fields. ◻

Space of Deviators. The space of bidirectional anisotropy vectors is closely related to the space of deviatoric tensors $\mathscr{D}$. A deviator D is the traceless or anisotropic part of a tensor T:

$$D = T - \frac{tr(T)}{2} I, \tag{7}$$

where I represents the unit tensor. The space of 2D symmetric deviatoric tensors $\mathscr{D}$ is a subspace of the set of 2D symmetric tensors $\mathscr{T}$ (see Fig. 3). The eigenvectors of D coincide with the eigenvectors of T. Thus the deviator field has the same topology as the original tensor field. Its eigenvalues are $\delta_1 = -\delta_2 = \frac{1}{2}(\lambda_1 - \lambda_2)$. The most commonly used norm in $\mathscr{T}$ is the Frobenius norm. For the deviator, the Frobenius norm is $\|D\|_F = \frac{1}{2}|\lambda_1 - \lambda_2|$, which corresponds to an anisotropy measure (shear stress) that is typically used for failure analysis in mechanical engineering and will be used as the anisotropy measure A below, that is, let $A = |\lambda_1 - \lambda_2|$. Based on the Frobenius norm, degenerate points are the points x_0 at which $\|D(x_0)\|_F = 0$. The Frobenius norm therefore induces a metric on $\mathscr{D}$, that is, for $D, D' \in \mathscr{D}$:

$$d_F(D, D') = \|D - D'\|_F. \tag{8}$$

Deviator and Bidirectional Vectors. If we restrict the mapping $\Omega|$ defined in Eq. (5) to the space of deviatoric tensors $\mathscr{D}$, the resulting mapping $\Omega|_{\mathscr{D}}$ is one-to-one. The inverse mapping is then defined by

$$\begin{aligned} &(\Omega|_{\mathscr{D}})^{-1} : \mathbb{P}^1 \times \mathbb{R}_{\geq 0} \to \mathscr{D} \\ &\qquad \omega = [Ae] \mapsto D = \frac{\sqrt{A}}{2}\ e \otimes e - \frac{\sqrt{A}}{2}\ e_\perp \otimes e_\perp. \end{aligned} \tag{9}$$

Here $e_\perp$ represents a normalized vector orthogonal to e. It can be seen immediately that this expression is independent on the sign of the representative vector e and thus is well-defined. For $\omega = [0e_0]$, Eq. 9 results in a zero tensor that is independent on the chosen vector e_0.

Theorem 3. *For the above defined metric Eq. (6) on the space of bidirectional anisotropy vectors and the Frobenius metric Eq. (8) on the space of deviators, we have*

$$\Omega(D') \in B_r(\Omega(D)) \Rightarrow D' \in B_{r'}(D) \tag{10}$$

with $r' = \sqrt{5}r$. For the opposite direction, we have

$$D' \in B_{r'}(D) \Rightarrow \Omega(D') \in B_r(\Omega(D)). \tag{11}$$

Thus the mapping defined in Eq. (5) is continuous, and the space of tensor deviators and the bidirectional anisotropy vectors are topologically equivalent.

Proof. Let D and D' be two symmetric, traceless 2D tensors with major eigenvalues $(\frac{1}{\sqrt{2}}\lambda, \frac{-1}{\sqrt{2}}\lambda)$ and $(\frac{1}{\sqrt{2}}\mu, \frac{-1}{\sqrt{2}}\mu)$, respectively, as well as their corresponding eigenvectors $[e_i]$ and $[f_i]$, for $i = 1,2$. Their norms are given by $\|D\|_F^2 = \lambda^2$ and $\|D'\|_F^2 = \mu^2$. The corresponding bidirectional anisotropy vectors are defined as $\omega = \Omega(D) = [Ae_1]$ with $A = \lambda$ and $\omega' = \Omega(D') = [A' f_1]$ with $A' = \mu$.

In order to compare the Frobenius distance between deviators and the distance between bidirectional anisotropy vectors, we first bring them into similar forms. Therefore we decompose the distance into two parts (see Appendix C for a derivation).

$$\begin{aligned} d_F^2(D,D') &= \|D - D'\|_F^2 = \|D\|_F^2 + \|D'\|_F^2 - 2(D : D') \\ &= (\lambda - \mu)^2 + 4\lambda\mu \, \sin^2 \alpha \end{aligned} \tag{12}$$

where $D : D'$ is the inner product of the tensors and α is the angle between the major eigenvectors. One can interpret this decomposition as having a shape-related part $(\lambda - \mu)^2$ and a direction-related part $4\lambda\mu \, \sin^2 \alpha$. A similar decomposition has been proposed by Zhang et al. [29] for the comparison of normalized tensors.

The distance defined between the bidirectional anisotropy vectors (Eq. (6)) is based on the L_2 distance between vectors. Therefore we will now express the L_2 distance between two vectors with length A and A' accordingly (see Appendix C for a derivation):

$$\begin{aligned} d^2(\omega, \omega') &= (A - A')^2 + 4A\,A' \, \sin^2(\alpha/2) \\ &= (\lambda - \mu)^2 + 4\lambda\mu \, \sin^2(\alpha/2). \end{aligned} \tag{13}$$

As in Eq. 12, we can interpret Eq. 13 as having a shape-related part and a distance-related part. The shape-related parts in Eq. (12) and Eq. (13) are identical; however, the direction-related parts differ with respect to the angels.

Now let $\Omega(D') \in B_r(\Omega(D))$ be a bidirectional vector in the r-ball of $\Omega(D)$ for some value $r \in \mathbb{R}_{>0}$, which means

$$d^2(\Omega(D), \Omega(D')) = (\lambda - \mu)^2 + 4\lambda\mu \, \sin^2(\alpha/2) \leq r^2.$$

It follows that $(\lambda - \mu)^2 \leq r^2$ and $4\lambda\mu \, \sin^2(\alpha/2) \leq r^2$. From this we can derive an upper limit for the Frobenius distance of the two tensors D' and D. Combining the relation $\sin\alpha = 2\sin(\alpha/2)\cos(\alpha/2)$ and the fact that $\alpha \in [0, \pi/2]$, we have $\sin\alpha \leq 2\sin(\alpha/2)$. It follows for the deviators:

$$d_F^2(D,D') = \underbrace{(\lambda - \mu)^2}_{\leq r^2} + \underbrace{4\lambda\mu \, \sin^2 \alpha}_{\leq 4\lambda\mu(4\sin^2(\alpha/2)) \leq 4r^2} \leq 5r^2 \Rightarrow D' \in B_{\sqrt{5}r}(D).$$

The opposite direction is trivially satisfied, since $\sin^2(\alpha/2) \leq \sin^2 \alpha$ for all $\alpha \in [0, \pi/2]$, and $d^2(\Omega(D), \Omega(D')) \leq d_F^2(D,D')$. ◻

Bidirectional Anisotropy Vector Field. In accordance with the tensor field, we now define a bidirectional anisotropy vector field.

A *bidirectional anisotropy vector field* $\boldsymbol{\omega}$ assigns to each position $x = (x_1, x_2) \in \mathbb{R}^2$ a bidirectional anisotropy vector ω. The map Ω can be used to convert the tensor field $\mathbf{T}$ into a bidirectional anisotropy vector field $\boldsymbol{\omega}(x) = \Omega(\mathbf{T}(x)) = (\Omega \circ \mathbf{T})(x)$. If the tensor field is continuous, then the bidirectional anisotropy vector field is also continuous, as it is a concatenation of two continuous mappings.

4.3 The Anisotropy Vector Field

In the following section, we define an *anisotropy vector field* $\tilde{\boldsymbol{\omega}}$ as a mapping from $\mathbb{R}^2$ to $\mathbb{S}^1 \times \mathbb{R}_{\geq 0}$. An element in $\mathbb{S}^1 \times \mathbb{R}_{\geq 0}$ can be understood as a vector in $\mathbb{R}^2$ represented in polar coordinates. Such a vector field $\tilde{\boldsymbol{\omega}}$ serves two purposes. First, we use it to specify the perturbation of a tensor field. Second, we use it to define the tensor index following the degree theory.

To define anisotropy vectors, we first define a mapping $\tilde{\Omega}$ from the space of tensors $\mathscr{T}$ to $\mathbb{S}^1 \times \mathbb{R}_{\geq 0}$ by lifting the first part of the mapping Ω from $\mathbb{P}^1$ to its covering space $\mathbb{S}^1$ using the mapping $\phi : \mathbb{P}^1 \to \mathbb{S}^1$ defined in Section 4.1. According to Eq. (5), we define

$$\begin{aligned} \tilde{\Omega} : \mathscr{T} &\to \mathbb{S}^1 \times \mathbb{R}_{\geq 0} \\ T &\mapsto \tilde{\Omega}(T) = \tilde{\omega} = ((\phi \times Id) \circ \Omega) = A e_1^2. \end{aligned} \tag{14}$$

Here $e_1 \in \mathbb{C}$ is an eigenvector representative of $[e_1]$ considered as a complex number. It can be easily seen that $Ae_1^2 = A(-e_1)^2$ is independent of the choice of the representative.

Theorem 4. *The above defined mapping (Eq. 14) restricted to the space of deviators $\tilde{\Omega}|_{\mathscr{D}}$ is an isometry with respect to the L_2-norm in $\mathbb{R}^2$ and the Frobenius norm in $\mathscr{D}^2$.*

Proof. The proof follows directly from Eq. (12) and (13). Let D and D' be two symmetric, traceless 2D tensors defined as above. We have

$$d^2(\tilde{\Omega}(D), \tilde{\Omega}(D')) = (\lambda - \mu')^2 + 4\lambda\mu \, \sin^2((2\alpha)/2) = d_F^2(D, D'),$$

since squaring a complex number doubles the angle. ◻

We would like to point out that thus defined vectors are less appropriate for geometric representations of the tensor and their directions are not directly correlated to the principal directions of the tensor. The explicit direction depends on the frame of reference chosen for the representation of the complex numbers (see also Section 4.4).

Anisotropy Vector Field. With these definitions, we can define the *anisotropy vector field* $\tilde{\boldsymbol{\omega}}$, which serves as basis for the application of the concept of robustness to tensor fields. $\tilde{\boldsymbol{\omega}}$ assigns to each position $x \in \mathbb{R}^2$ a vector $\tilde{\omega}$. Thereby the map $\tilde{\Omega}$ is used to convert the tensor field $\mathbf{T}$ into a vector field $\tilde{\boldsymbol{\omega}}(x) = \tilde{\Omega}(\mathbf{T}(x)) = (\tilde{\Omega} \circ \mathbf{T})(x)$. If the tensor field is continuous, then the anisotropy vector field is also continuous as a concatenation of two continuous mappings (see Fig. 5).

$$\tilde{\boldsymbol{\omega}} : \mathbb{R}^2 \to \mathbb{S}^1 \times \mathbb{R}_{\geq 0}(\simeq \mathbb{R}^2), \tilde{\boldsymbol{\omega}}(x) = ((\phi \times Id) \circ \boldsymbol{\omega})(x) \tag{15}$$

Therefore $Id : \mathbb{R} \to \mathbb{R}$ is the identity map.

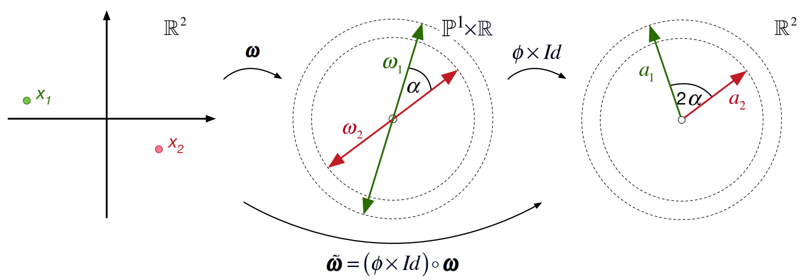

Fig. 5: The concatenation of the mapping defined by the tensor field and the homomorphism ϕ between $\mathbb{P}$ and $\mathbb{S}$ defined in Section 4.1 is a continuous mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. It defines a vector field on $\mathbb{R}^2$.

4.4 Notes on the Topology of the Anisotropy Vector Field

When looking at the vector field derived from the tensor field in Section 4.3, an obvious question is how its vector field topology relates to the tensor topology of the original tensor field. From the construction of the anisotropy vector field, it is clear that its critical points, zeros of the vector field, coincide with the degenerate points of the tensor field.

These points, however, constitute only a part of the topology. The second essential part is the connecting separatrices. For the vector case, these are the integral lines of the vector field. For tensor fields, the separatrices are tensor lines, which follow one eigenvector field. The structure in the vicinity of the critical points is characterized by its index (compare to Section 5.1). In our setting, the index of the tensor field degenerate points and the index of the anisotropy vector field are related by the degree two mapping ϕ defined in 4.1. Thus a wedge point in the tensor field (tensor index $+1/2$) is mapped to sources/sinks (vector index $+1$) and trisectors (tenors index $-1/2$) are mapped to saddle points (vector index -1). In general a degenerate point of tensor index i is mapped to a critical point of index $2i$ (Fig. 6). This mapping gives rise to a very distinct structure in which different critical points

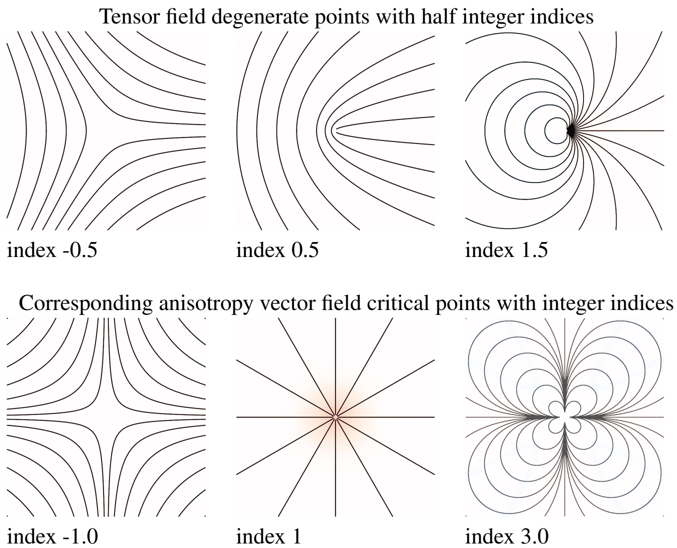

Fig. 6: Change of the structure of the field when mapping the tensor field to the anisotropic vector field. Examples for isolated degenerate points.

will be connected. Integral lines in the vector field do not coincide with the integral lines for the tensor field.

What is important, however, for our discussion is that the stability of the critical points and the degenerated points in terms of robustness is the same.

5 Robustness for Tensor Fields

Similar to vector field topology, one of the major challenges in tensor field topology is the complexity of the topological structure. A large part of the topological structure originates from extended isotropic regions and such a structure is very sensitive to small changes in the data. Therefore we would like to have a *stable* topological skeleton representing the core structure of the data. Previous attempts to simplify the tensor field topology have relied on heuristics that lack a clean mathematical framework. Motivated by the notion of robustness based on the well group theory for the vector fields, we extend such a concept to 2D symmetric second-order tensor fields. In this section, we connect the indexes of degenerate points with the degree theory in Section 5.1, define tensor field perturbation in Section 5.2, and generalize robustness to tensor field topology in Section 5.3. Our main contributions are three-fold: We interpret the notion of tensor index under the setting of degree theory; We define tensor field perturbations and make precise connections between such perturbations with the perturbations of bidirectional anisotropy vector fields; And we generalize the notion of robustness to tensor field topology.

5.1 Indexes of Degenerate Points and Degree Theory

Index of Degenerate Points. Similar to the zeros of vector fields, we also consider the notion of index for these degenerate points. Delmarcelle [9] defines the index of a degenerate point $x \in \mathbb{R}^2$ as the number of "half-windings" an eigenvector performs when moved along a simple closed curve (i.e., a Joran curve) enclosing the degenerate point. For linear fields, the structure of the eigenvector fields surrounding the degenerate points follows two characteristic patterns depending on their indexes (see Fig. 7).

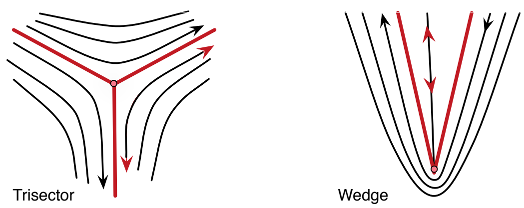

Fig. 7: Basic structure of eigenvector fields in the vicinity of degenerate points. It can easily be seen that it is not possible to orient the tensor lines in a continuous way.

Connection to Degree Theory. The above definition of an index by Delmarcelle follows a geometric point of view considering the number of "half-windings" of the eigenvectors. There is also, however, a close connection between the index of a degenerate point and the degree of a mapping as defined in algebraic topology. Since the degree plays an important role in the theory of robustness, we revisit the concept in Section 3 and provide here a formulation in terms of degree theory. The line of thought is similar to that of Trichoche ([25], page 55).

Consider a tensor field T defined on an orientable surface $\mathbb{M}$ (here $\mathbb{M} = \mathbb{R}^2$), and suppose all degenerate points are isolated and finite in number. We would like to associate (via the theory of Hopf [14]) an index with each x of $\mathbb{M}$. We built a continuous mapping ξ by lifting the eigenvector field ψ to its covering space $\mathbb{S}^1$ (see Fig. 8). That is, $\xi = \phi \circ \psi$, where $\psi : \mathbb{R}^2 \to \mathbb{P}^1$ and $\phi : \mathbb{P}^1 \to \mathbb{S}^1$, that is, $\xi(x) = [e_1]^2$, where $[e_1]$ is a generator.

For the definition of the index of a degenerate point x, we consider the boundary of a region C enclosing x, that is, the curve ∂C with no other degenerate points in its interior. We define the index of x, $I_T(x)$ to be $\frac{1}{2}\deg(\xi|_{\partial C})$.

Poincaré-Hopf Theorem for Tensor Fields. Delmarcelle has provided a tensor field equivalence of Poincaré-Hopf theorem ([9], page 163).

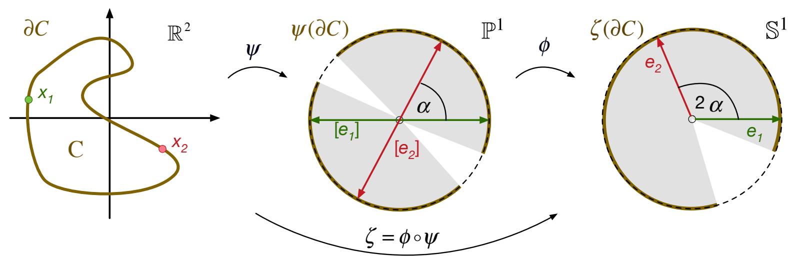

Fig. 8: The mapping $\xi = \phi \circ \psi$ defines a continuous mapping from $\mathbb{R}^2$ to $\mathbb{S}^1$, which corresponds to lifting the mapping ϕ from $\mathbb{P}^1$ to the covering space $\mathbb{S}^1$.

Theorem 5 (Theorem 15, [9]). *The tensor index of a 2D orientable surface $\mathbb{M}$ relative to a tangent tensor field T with a finite number of degenerate points on $\mathbb{M}$ is equal to the Euler characteristic of $\mathbb{M}$. That is,* $\mathrm{I}_T(\mathbb{M}) = \sum_i \mathrm{I}_T(x_i) = \chi(\mathbb{M})$.

According to Hopf's result [14], whenever the continuous field of directions tangent to $\mathbb{M}$ is not zero at more than finitely many points x_i, we always have the above theorem [5].

5.2 *r-Perturbation of Anisotropy Vector Field*

Suppose we have two anisotropy vector fields f and h, derived from tensor fields $\mathbf{T}$ and $\mathbf{T}'$, respectively, that is, $f = \tilde{\boldsymbol{\omega}}$ and $h = \tilde{\boldsymbol{\omega}}'$. We define the distance between the two as

$$\mathrm{d}(f,h) = \sup_{x \in \mathbb{R}^2} ||f(x) - h(x)||_2.$$

We say a continuous mapping h is an *r-perturbation* of f, if $\mathrm{d}(f,h) \leq r$. In other words, for each point $x \in \mathbb{R}^2$, the point $h(x)$ lies within a disk of radius r centered at $f(x)$. See Fig. 2 for a geometric interpretation of an r-perturbation of the anisotropy vector field at a point x in the domain.

5.3 *Robustness of Degenerate Points*

Converting a tensor field $\mathbf{T}$ to its corresponding anisotropy vector field f greatly simplifies the extension of robustness from the vector field to the setting of the tensor field. First, the degenerate points of $\mathbf{T}$ correspond to the critical points of f; therefore f has no critical points in a path-connected region $C \subset \mathbb{R}^2$ iff $\mathbf{T}$ has no degenerate points in C. Second, the index of a degenerate point in $\mathbf{T}$ is half the degree of its corresponding critical point in f. Third, the r-perturbation of f relates to the perturbation of $\mathbf{T}$ via its projection $\mathbf{D}$ in a quantifiable way; an r-perturbation in f corresponds to an r-perturbation in $\mathbf{D}$.

We have conjectured that the robustness of degenerate points x for tensor fields $\mathbf{T}$ would resemble the robustness of its corresponding critical point f for the anisotropy vector fields. Recall that, by definition, f is an anisotropy vector field, $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f_0 = ||f||_2 : \mathbb{R}^2 \to \mathbb{R}$, $\mathbb{F}_r = f_0^{-1}(-\infty, r]$. Let h be another anisotropy vector field $h : \mathbb{R}^2 \to \mathbb{R}^2$. We would prove the following lemmas, whose proofs are identical to the proofs used for results in [28] (Corollary 1.1 and Corollary 1.2 in the supplemental material) with respect to vector field perturbation. We include the proofs here for completeness.

Lemma 4 (Nonzero Degree Component for Tensor Field Perturbation). *Let r be a regular value of f_0 and C a connected component of $\mathbb{F}_r$ such that* $\deg(C) \neq 0$. *Then for any δ-perturbation h of f, where $\delta < r$, the sum of the degrees of the critical points in $h^{-1}(0) \cap C$ is* $\deg(C)$.

Proof. Before we illustrate the details of the proof, we need to provide a rigorous definition of the degree of a mapping.

Let $C \subseteq \mathbb{F}_r$ be a path-connected component of $\mathbb{F}_r$. Function f restricted to C, denoted $f|_C : (C, \partial C) \to (B_r, \partial B_r)$, maps C to the closed ball B_r of radius r centered at the origin, where ∂ is the boundary operator. $f|_C$ induces a homomorphism on the homology level, $f_*|_C : \mathsf{H}(C, \partial C) \to \mathsf{H}(B_r, \partial B_r)$. Let μ_C and μ_{B_r} be the generators of $\mathsf{H}(C, \partial C)$ and $\mathsf{H}(Br, \partial B_r)$, respectively. The *degree* of C (more precisely the degree of $f|_C$), $\deg(C) = \deg(f|_C)$, is the unique integer such that $f_*|_C(\mu_C) = \deg(C) \cdot \mu_{B_r}$. Furthermore we have the function restricted to the boundary, that is, $f|_{\partial C} : \partial C \to \mathbb{S}^1$. It was shown that $\deg(f|_C) = \deg(f|_{\partial C})$ ([7], Lemma 1).

Consider the following diagram for any δ-perturbation h of f, where $\delta < r$:

$$\begin{array}{ccc} \mathsf{H}(C, \partial C) & \xrightarrow{i_*} & \mathsf{H}(C, C - h^{-1}(0)) \\ \downarrow f_*|_C & & \downarrow h_*|_0 \\ \mathsf{H}(B_r, \partial B_r) & \xrightarrow{j_*} & \mathsf{H}(B_r, B_r - \{0\}). \end{array} \quad (16)$$

i_* and j_* are homomorphisms induced by space-level inclusions $i : (C, \partial C) \to (C, C - h^{-1}(0))$ and $j : (B_r, \partial B_r) \to (B_r, B_r - \{0\})$. j_* is also an isomorphism. The vertical maps $f_*|_C$ and $h_*|_0$ are induced by f and h with restrictions, respectively. Therefore the diagram commutes.

Suppose r is a regular value and $\deg(C) \neq 0$. Then by commutativity, the sum of degrees of the critical points in $h^{-1}(0) \cap C$ is $\deg(C)$. ⊔⊓

Lemma 5 (Zero Degree Component for Tensor Field Perturbation). *Let r be a regular value of f_0 and C a connected component of $\mathbb{F}_r$ such that* $\deg(C) = 0$. *Then there exists an r-perturbation h of f such that h has no degenerate points in C, $h^{-1}(0) \cap C = \emptyset$. In addition, h equals f except possibly within the interior of C.*

Proof. The proof follows the commutative diagram above (Eq. 16) for any r-perturbation h of f. Suppose r is a regular value. Then well groups $\mathsf{U}(r - \delta)$ and $\mathsf{U}(r + \delta)$ are isomorphic for all sufficiently small $\delta > 0$. Suppose $\deg(C) =$

$\deg(f|_C) = \deg(f|_{\partial C}) = 0$. Then following the Hopf Extension Theorem ([13], page 145), if the function $f|_{\partial C} : \partial C \to \mathbb{S}^1$ has degree zero, then f can be extended to a globally defined map $g : C \to \mathbb{S}^1$ such that g equals f when both are restricted to ∂C. Now we define a perturbation $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that $h = 0.5 \cdot f + 0.5 \cdot g$. h is the midpoint on a straight line homotopy between f and g. By definition $\mathrm{d}(h, f) \leq r$, so h is an r-perturbation of f. In addition, $h^{-1}(0) \cap C$ is empty. □

Remark. One important aspect of well group theory is that the well group is defined to be the intersection of the images of j_h for *all* r-perturbation h of f (Eq. 1). Given f as an anisotropy vector field, we introduce an r-perturbation h of f. We would need to make sure that *any* such h is itself a valid anisotropy vector field. That is, for any r-perturbation h of f, there exists a corresponding tensor field **T** from which an anisotropy vector field h can be derived. This is true based on derivations in Section 4.

6 Discussion

There are a few challenges in extending our framework to a 3D symmetric tensor field. The notion of deviator can be generalized to 3D, but the notion of anisotropy vector field does not generalize to 3D. The lack of such a notion poses a challenge in studying robustness for 3D symmetric tensor field topology via transformation of the data to the anisotropy vector field. We suspect a possible solution is to define perturbations with respect to the bidirectional anisotropy vector field derived from eigenvector fields.

An important contribution of this paper is the conversion from a tensor field **T** to its corresponding anisotropy vector field f. There is a one-to-one correspondence between the degenerate points of **T** and the critical points of f. However, as shown in Fig. 6, the topology of **T** and that of f obviously do not agree. Understanding their differences and the consequences will be an interesting direction.

The main motivation of extending robustness to 2D symmetric tensor field is that it would lead to simplification schemes for tensor field data. In general, topology-based simplification techniques pair the topological features for simplification via the computation of topological skeleton, which can be numerically unstable. In contrast, the proposed robustness-based method is independent of the topological skeleton and, thus, is insensitive to numerical error.

Appendix A Notations

Symbol	Description
$\mathscr{T}$	Space of 2D symmetric second-order tensors
$\mathscr{D} \subset \mathscr{T}$	Space of 2D symmetric second-order deviators
$\mathbb{RP}^1$ or $\mathbb{P}^1$	Real projective line
T, D	Tensors
$\mathbf{T}, \mathbf{D} : \mathbb{R}^2 \to \mathscr{T}$	Tensor fields
v_i	Eigenvectors
e_i	Normalized (unit) eigenvectors
λ_i, μ_i	Eigenvalues
A (e.g. $= \lvert\lambda_1 - \lambda_2\rvert$)	Anisotropy measure
$[e_i] \in \mathbb{P}^1$	Equivalence class of unit eigenvectors
$\boldsymbol{\psi}_i : \mathbb{R}^2 \to \mathbb{P}^1$, $i = 1, 2$,	Major ($i = 1$) and minor ($i = 2$) eigenvector fields (direction fields, no magnitude)
$\boldsymbol{\psi} = \boldsymbol{\psi}_1 : \mathbb{R}^2 \to \mathbb{P}^1$	(Major) eigenvector field
ω	Bidirectional anisotropy vector
$\Omega : \mathscr{T} \to \mathbb{P}^1 \times \mathbb{R}_{\geq 0}$	Mapping that assigns a bidirectional anisotropy vector to the tensor
$\boldsymbol{\omega} : \mathbb{R}^2 \to \mathbb{P}^1 \times \mathbb{R}_{\geq 0}$	Bidirectional anisotropy vector field (direction fields with magnitude, not a traditional vector field)
$\tilde{\Omega} : \mathscr{T} \to \mathbb{R}^2$	Mapping that assigns an anisotropy vector to a tensor
$\tilde{\boldsymbol{\omega}} : \mathbb{R}^2 \to \mathbb{R}^2$	Anisotropy vector field (traditional vector field)
$\tilde{\omega}(x) \in \mathbb{R}^2$	Anisotropy vector
$\phi : (\mathbb{P}^1 \simeq (\mathbb{S}^1 / \sim)) \to \mathbb{S}^1$	Degree 2 mapping
$\xi = \phi \circ \psi : \mathbb{R}^2 \to \mathbb{S}^1$	Mapping used for defining the index of degenerate points

Appendix B Triangle Inequality for the Distance Measure Between Bidirectional Anisotropy Vectors (Eq. 6)

Let ω, ω' and ω'' be bidirectional anisotropy vectors as defined in Eq. (5) with vector representatives v, w, and u respectively. Recall the distance measure is defined as $d(\omega, \omega') = \min(\|v - w\|_2, \|v + w\|_2)$. Therefore we have:

$$
\begin{aligned}
&\mathrm{d}(\omega,\omega')+\mathrm{d}(\omega',\omega'')\\
&\quad = \min(\quad \|v-w\|_2, \|v+w\|_2) + \min(\|w-u\|_2, \|w+u\|_2)\\
&\quad \geq \min(\quad \|v-w\|_2+\|w-u\|_2, \|v-w\|_2+\|w+u\|_2,\\
&\qquad\qquad\quad \|v+w\|_2+\|w-u\|_2, \|v+w\|_2+\|w+u\|_2\ )\\
&\quad = \min(\quad \|v-w\|_2+\|w-u\|_2, \|v-w\|_2+\|w-(-u)\|_2,\\
&\qquad\qquad\quad \|v-(-w)\|_2+\|(-w)-(-u)\|_2, \|v-(-w)\|_2-\|(-w)-u\|_2\ )\\
&\quad \geq \min(\quad \|v-u\|_2, \|v-(-u)\|_2, \|v-(-u)\|_2, \|v-u\|_2\ )\\
&\quad = \min(\quad \|v-u\|_2, \|v+u\|_2\ )\\
&\ = \mathrm{d}(\omega,\omega'')
\end{aligned}
$$

Appendix C Derivations for Eq. 12 and Eq. 13

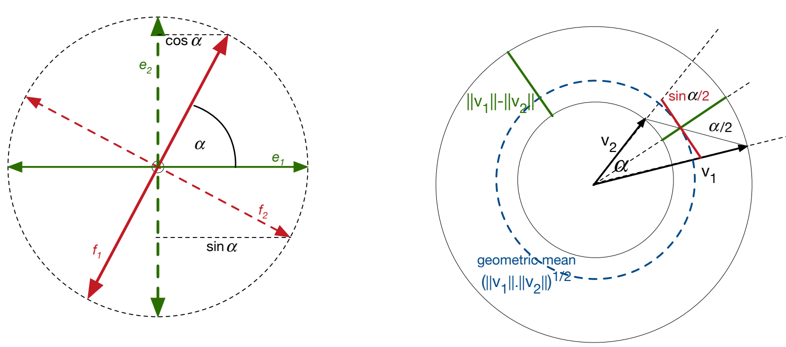

Fig. 9: Left: eigenvectors of D and D' and angle α by definition. Right: geometric interpretation of the vector distance decomposed in radial and directional parts.

The inner product of two symmetric tensors T and T' is defined as $T : T' = \sum_{ij} t_{ij} t'_{ij}$. It can be expressed in terms of eigenvectors and eigenvalues $\sum_{ks} \lambda_k \mu_s (e_k \cdot f_s)^2$ (see e.g., [4]). Here $\cdot$ denotes the standard scalar product of vectors. For 2D deviatoric tensors D and D' with eigenvalues $(\frac{1}{\sqrt{2}}\lambda, \frac{-1}{\sqrt{2}}\lambda)$ and $(\frac{1}{\sqrt{2}}\mu, \frac{-1}{\sqrt{2}}\mu)$, respectively, and their corresponding eigenvectors e_i and f_i (for $i = 1,2$), this yields

$$(D:D') = \frac{1}{2}\lambda\mu\left((e_1\cdot f_1)^2-(e_1\cdot f_2)^2-(e_2\cdot f_1)^2+(e_2\cdot f_2)^2\right) = \lambda\mu\left(\cos^2\alpha-\sin^2\alpha\right)$$

$$\begin{aligned} d_F^2(D,D') = \|D-D'\|_F^2 &= \|D\|^2 + \|D'\|^2 - 2(D:D') \\ &= \|D\|^2 + \|D'\|^2 - 2\lambda\mu\left(\cos^2\alpha - \sin^2\alpha\right) \\ &= \lambda^2 + \mu^2 - 2\lambda\mu\left(1 - 2\sin^2\alpha\right) \\ &= (\lambda-\mu)^2 - 4\lambda\mu\sin^2\alpha \end{aligned}$$

A similar construction for 2D vectors v_1 and v_2 using the trigonometric equality $1-\cos(2\beta) = 2\sin^2(\beta)$ gives:

$$\begin{aligned} d^2(v_1,v_2) &= \|v_1\|^2 + \|v_2\|^2 - 2(v_1\dot{v}_2) \\ &= (\|v_1\| + \|v_2\|)^2 - 2\|v_1\|\,\|v_2\| - 2(v_1\cdot v_2) \\ &= (\|v_1\| + \|v_2\|)^2 - 2\|v_1\|\,\|v_2\|\,(1-2\cos\alpha) \\ &= (\|v_1\| + \|v_2\|)^2 - 4\|v_1\|\,\|v_2\|\sin^2(\alpha/2) \end{aligned}$$

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