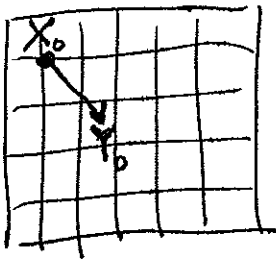
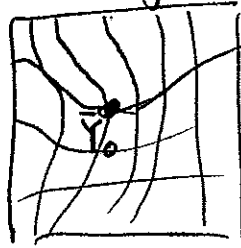


Warping with Radial Basis Functions RBF's

source



target



(\bar{X}_0, \bar{Y}_0) pair of
landmarks
in source and
target image

concept: → find transformation so that:

- exact transformation at landmarks
- smooth interpolation in neighborhood

$$\boxed{T(\bar{X}) = \bar{X} + \bar{v}(\bar{X}) = \bar{Y}}$$

- radial basis functions:

$$\phi(\|\bar{X} - \bar{Y}\|)$$

possible choices: $\phi() = \exp\left(-\frac{\|\bar{X} - \bar{Y}\|^2}{\delta^2}\right)$

$$\phi() = \frac{\delta^2}{\|\bar{X} - \bar{Y}\|^2 + \delta^2}$$

...

important: functions have maximum value
for $\bar{X} = \bar{Y}$, and gradually fall off
with increasing radial distance

- use radial basis functions
centered at landmarks \bar{X}_0

Example: 2 landmark pairs

$$\begin{aligned} (\bar{X}_1, \bar{Y}_1) &\rightarrow \left[\begin{pmatrix} X_1^x \\ X_1^y \end{pmatrix}, \begin{pmatrix} Y_1^x \\ Y_1^y \end{pmatrix} \right] \\ (\bar{X}_2, \bar{Y}_2) &\rightarrow \left[\begin{pmatrix} X_2^x \\ X_2^y \end{pmatrix}, \begin{pmatrix} Y_2^x \\ Y_2^y \end{pmatrix} \right] \end{aligned}$$

$$T(\bar{X}) = \bar{X} + \sum_{i=1}^2 \bar{k}_i \phi(\|\bar{X} - \bar{X}_i\|)$$

component notation for estimation of "weights" \bar{k}_i :

$$\begin{aligned} \begin{cases} T^x(\bar{X}_1) = X_1^x + k_1^x \phi(\|\bar{X}_1 - \bar{X}_1\|) + k_2^x \phi(\|\bar{X}_1 - \bar{X}_2\|) = Y_1^x \\ T^y(\bar{X}_1) = X_1^y + k_1^y \phi(\|\bar{X}_1 - \bar{X}_1\|) + k_2^y \phi(\|\bar{X}_1 - \bar{X}_2\|) = Y_1^y \end{cases} \\ \begin{cases} T^x(\bar{X}_2) = X_2^x + k_1^x \phi(\|\bar{X}_2 - \bar{X}_1\|) + k_2^x \phi(\|\bar{X}_2 - \bar{X}_2\|) = Y_2^x \\ T^y(\bar{X}_2) = X_2^y + k_1^y \phi(\|\bar{X}_2 - \bar{X}_1\|) + k_2^y \phi(\|\bar{X}_2 - \bar{X}_2\|) = Y_2^y \end{cases} \end{aligned}$$

$$\text{let } B = \begin{bmatrix} \phi(\bar{X}_1, \bar{X}_1) & \phi(\bar{X}_1, \bar{X}_2) \\ \phi(\bar{X}_2, \bar{X}_1) & \phi(\bar{X}_2, \bar{X}_2) \end{bmatrix} = \begin{bmatrix} 1 & \phi(\bar{X}_1, \bar{X}_2) \\ \phi(\bar{X}_2, \bar{X}_1) & 1 \end{bmatrix}^*$$

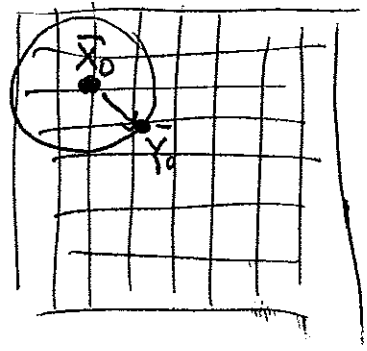
$$\begin{pmatrix} X_1^x \\ X_2^x \\ X_1^y \\ X_2^y \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} k_1^x \\ k_2^x \\ k_1^y \\ k_2^y \end{pmatrix} = \begin{pmatrix} Y_1^x \\ Y_2^x \\ Y_1^y \\ Y_2^y \end{pmatrix}$$

\Rightarrow 4 equations to solve for 4 unknowns

* B is symmetric since $\phi(\|\bar{X}_i - \bar{X}_j\|) = \phi(\|\bar{X}_j - \bar{X}_i\|)$

• let \bar{X}_0, \bar{Y}_0 be a set of landmarks

• $\phi(\cdot)$ centered at \bar{X}_0
 $\rightarrow \phi(\|\bar{X} - \bar{X}_0\|)$



• $T(\bar{X}) = \bar{X} + \underbrace{\bar{v}(\bar{X})}_{\text{displacement field}}$

$$= \bar{X} + \bar{k}_0 \cdot \phi(\|\bar{X} - \bar{X}_0\|) = \bar{Y}$$

\bar{k}_0 : weight 2D vector to be determined, called "momentum"

• (x, y) -components: $T^x(\bar{x}) = X^x + k_0^x \cdot \phi(\|\bar{X} - \bar{X}_0\|) = Y^x$
 $T^y(\bar{x}) = X^y + k_0^y \cdot \phi(\|\bar{X} - \bar{X}_0\|) = Y^y$

Estimation of \bar{k}_0 :

$$T(\bar{X}_0) = \bar{Y}_0$$

transformation exactly matches landmark deformation

$$\Rightarrow \bar{X}_0 + \underbrace{\bar{k}_0 \phi(\bar{X}_0 - \bar{X}_0)}_{\bar{v}(\bar{X}_0)} = \bar{Y}_0$$

$$\Rightarrow \bar{X}_0 + \bar{k}_0 \cdot \underbrace{\phi(0)}_1 = \bar{Y}_0$$

$$\Rightarrow \underline{\underline{\bar{k}_0 = (\bar{Y}_0 - \bar{X}_0)}}$$

$$\underline{\underline{T(\bar{X}) = \bar{X} + \bar{k}_0 \cdot \phi(\|\bar{X} - \bar{X}_0\|) = \bar{Y}}}$$

After the moments (\bar{k}_i) are known, the image deformation is finally: $T(\bar{X}) = \bar{X} + \bar{v}(\bar{X}) \Rightarrow$

$$\underline{\underline{T(\bar{X}) = \bar{X} + \sum_{i=1}^n \bar{k}_n \phi(\|\bar{X} - \bar{X}_n\|) = \bar{Y}}}$$

This is a forward transformation and as discussed in the course should rather be implemented as the inverse. However, the transformation may not have an inverse (see document Stanley Durkin, "lecture-10-18.pdf", page 10 for more details).

Approximative solution:

$$\begin{aligned} T^{-1}(\bar{Y}) &= \bar{Y} - \bar{v}(\bar{Y}) \\ &= \bar{Y} - \sum_{i=1}^n \bar{k}_n \phi(\|\bar{Y} - \bar{X}_n\|) \approx \bar{X} \end{aligned}$$

stepping through the target raster, one can approximately calculate the corresponding source pixel location and get the pixel values by interpolation (equivalent to linear transformations).