Filtering in the Fourier Domain

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Fourier Filtering

- Low-pass filtering
- High-pass filtering
- Band-pass filtering
- Sampling and aliasing
- Tomography
- Optimal filtering and match filters

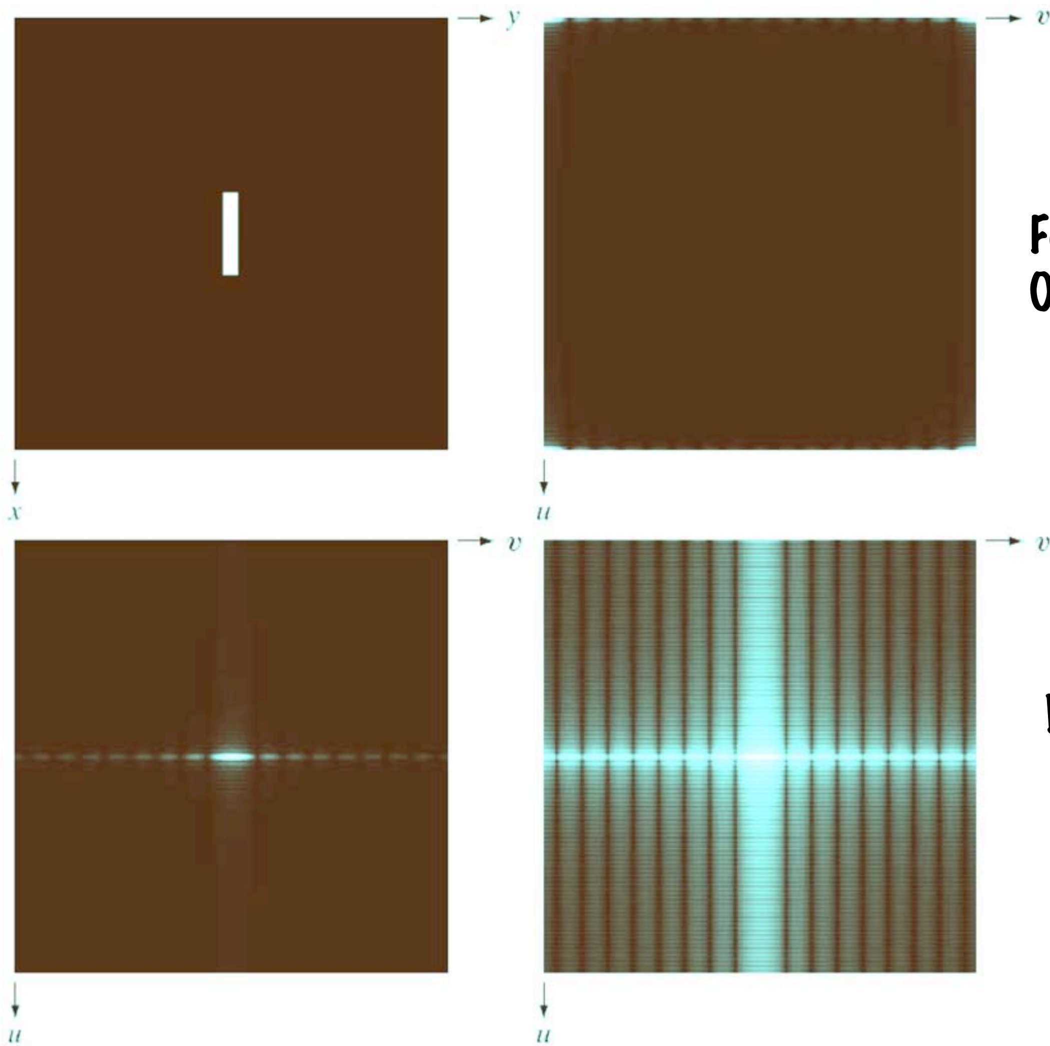
Some Identities to Remember

Discrete unit impulse	$\delta(x, y) \Leftrightarrow 1$
Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\frac{\sin(\pi ua)}{(\pi ua)}\frac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\frac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)\right]$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\frac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0)\right]$
Gaussian	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

Fourier Spectrum

Image

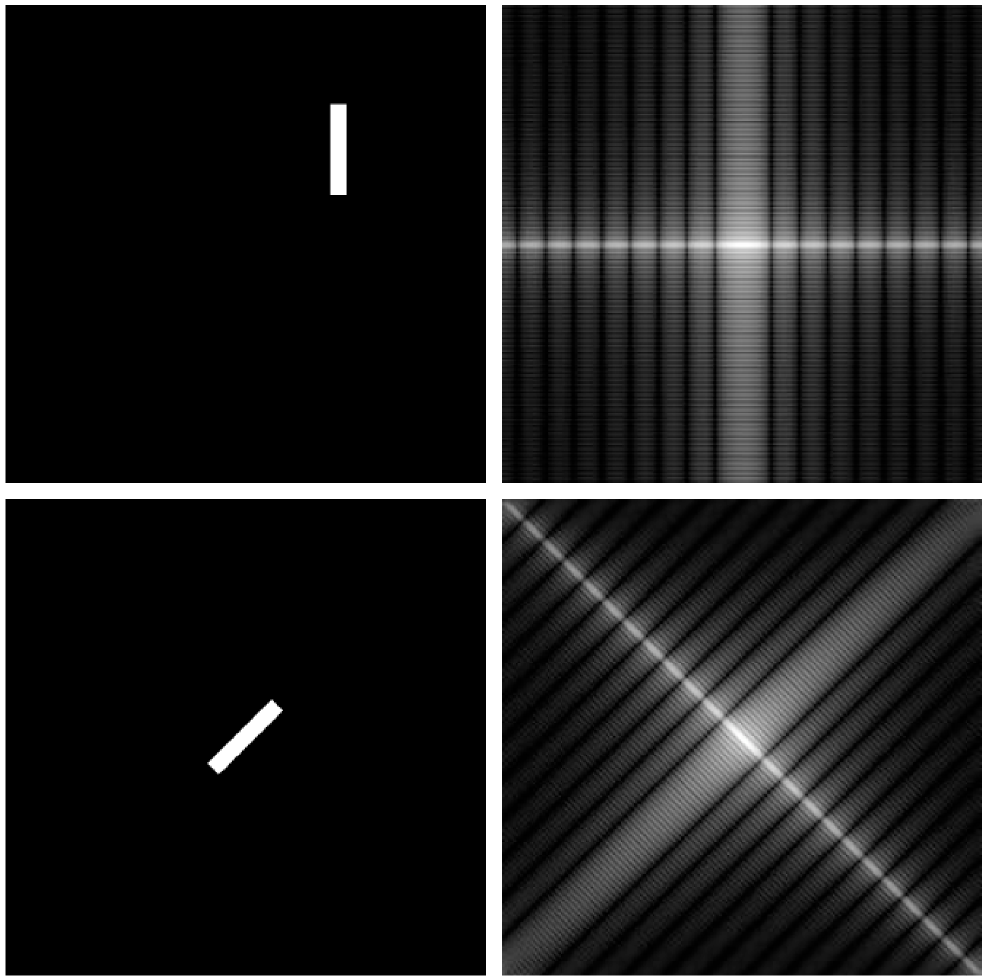
Fourier spectrum
Origin in corners



Retiled with origin
In center

Log of spectrum

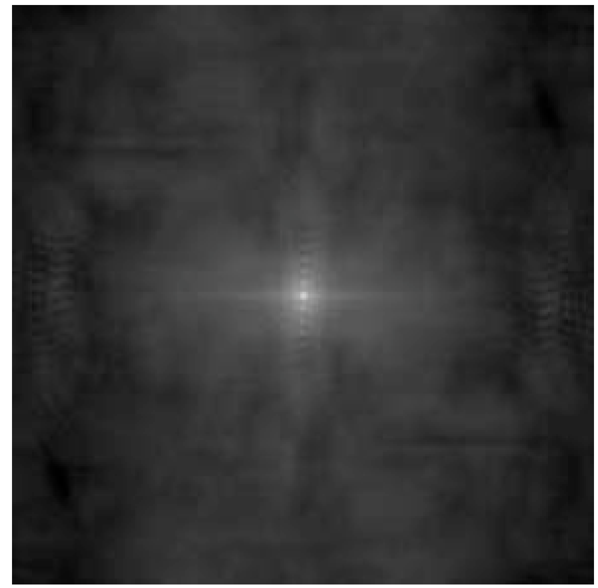
Fourier Spectrum-Rotation



Phase vs Spectrum

Image

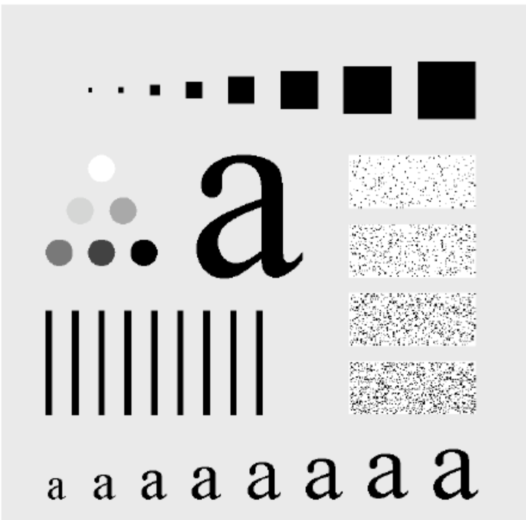
Reconstruction from
phase map

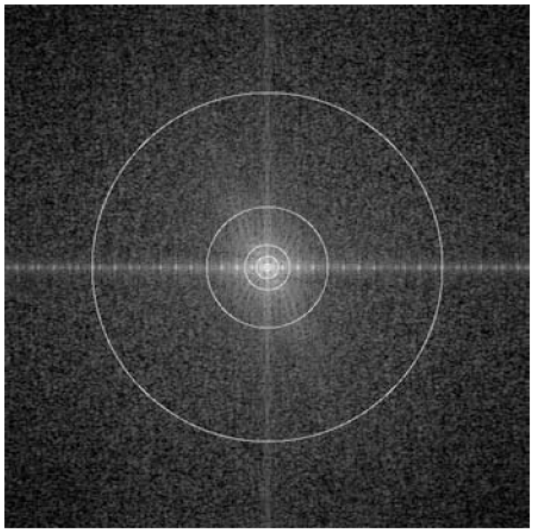


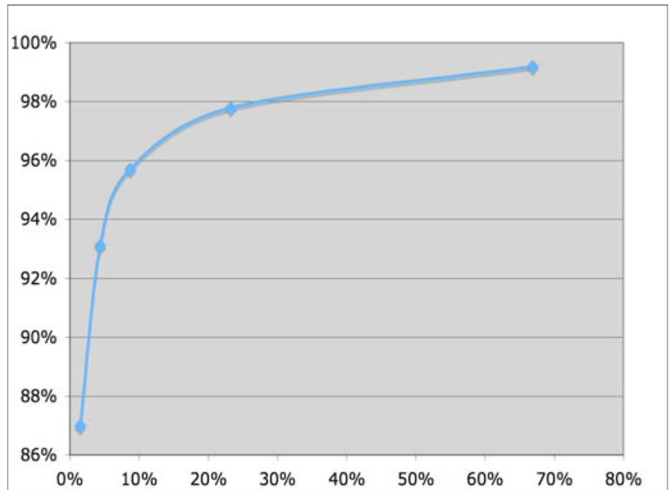
Reconstruction from
spectrum

Low-Pass Filter

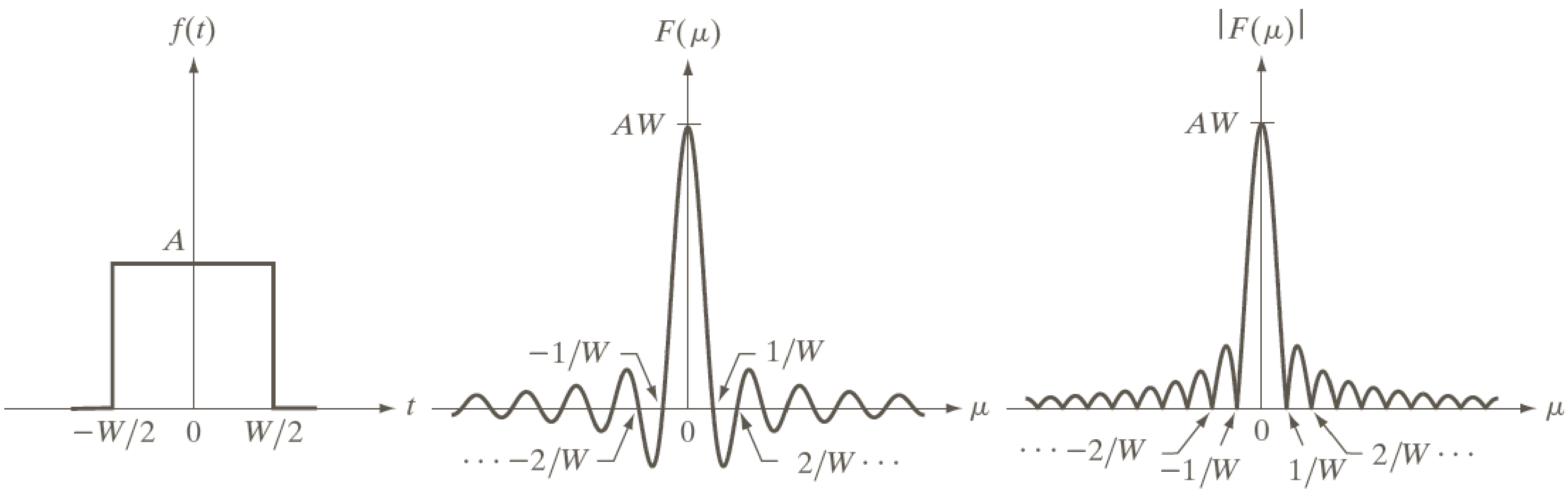
- Reduce/eliminate high frequencies
- Applications
  - Noise reduction
    - uncorrelated noise is broad band
    - Images have sprectrum that focus on low frequencies







Ideal LP Filter – Box, Rect

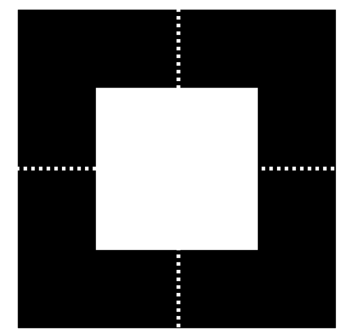


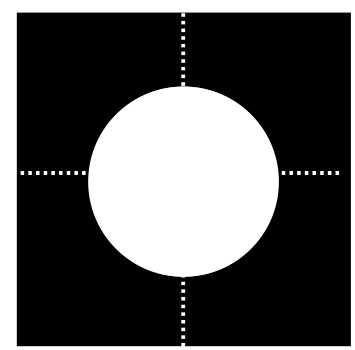
Cutoff freq

Ringing – Gibbs phenomenon

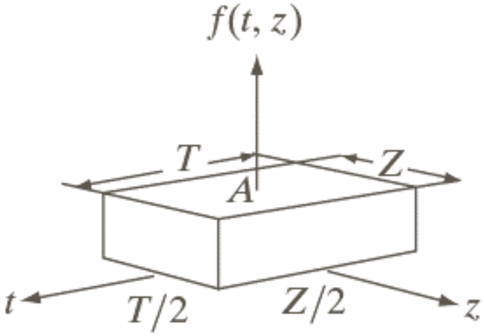
Extending Filters to 2D (or higher)

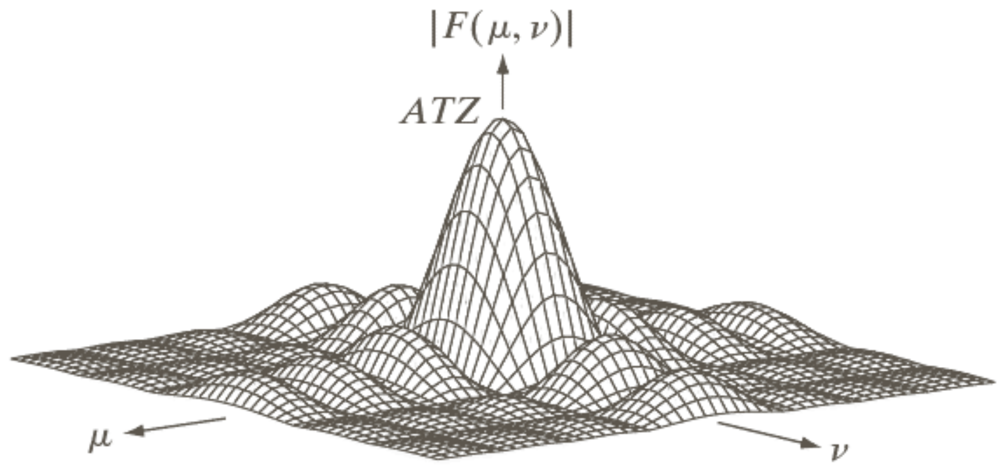
- Two options
  - Separable
    - $H(s) \rightarrow H(u)H(v)$
    - Easy, analysis
  - Rotate
    - $H(s) \rightarrow H((u^2 + v^2)^{1/2})$
    - Rotationally invariant





Ideal LP Filter - Box, Rect





Ideal Low-Pass
Rectangle With Cutoff of 2/3

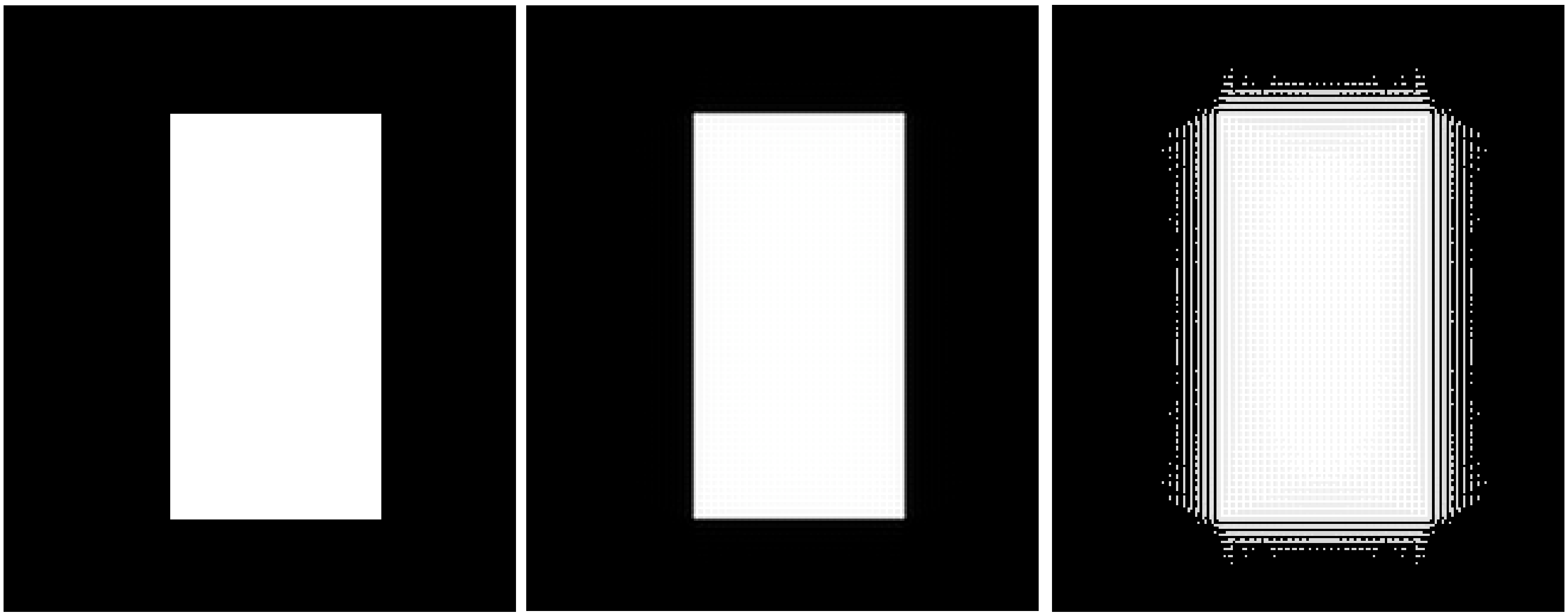


Image Filtered Filtered + HE

Ideal LP - 1/3

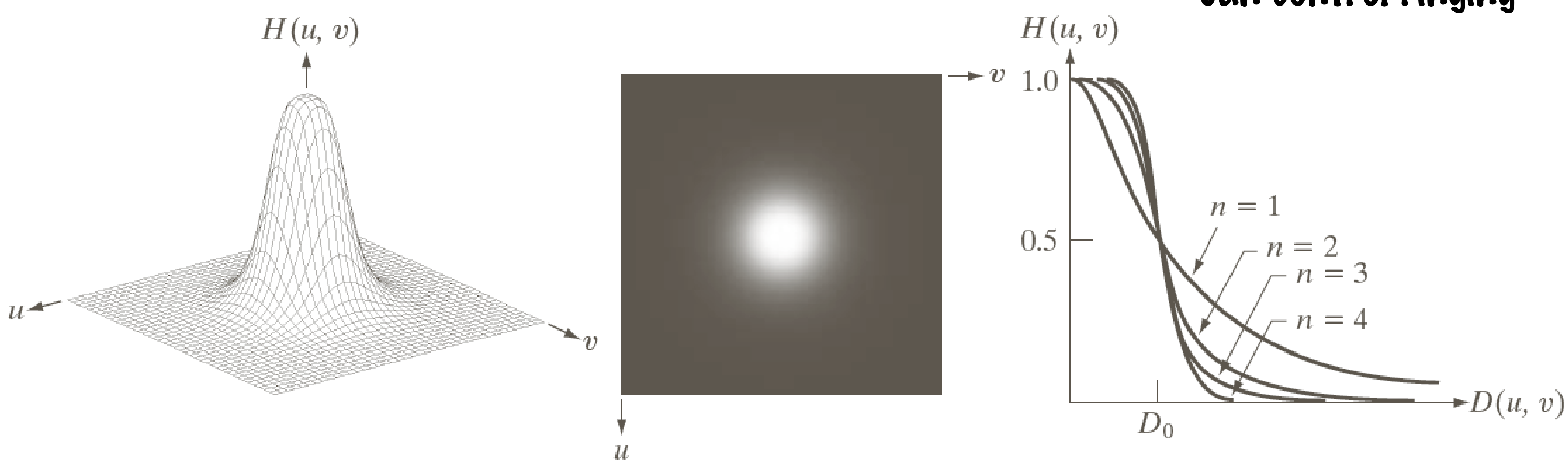

Ideal LP - 2/3

Butterworth Filter

Lowpass filters. D_0 is the cutoff frequency and n is the order of the Butterworth filter.

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$	$H(u,v) = \dfrac{1}{1 + [D(u,v)/D_0]^{2n}}$	$H(u,v) = e^{-D^2(u,v)/2D_0^2}$

Control of cutoff and slope
Can control ringing

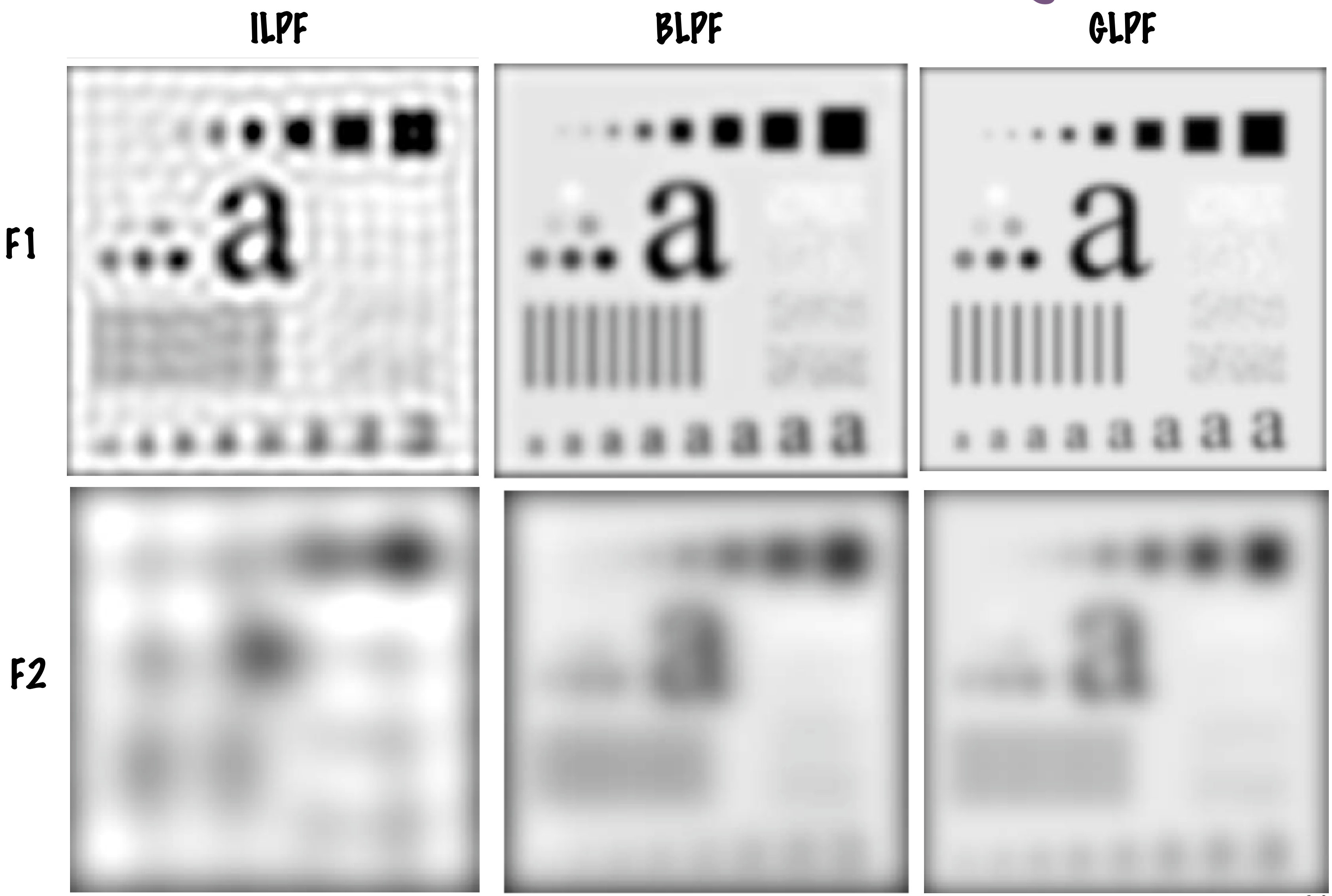


Butterworth - 1/3

Butterworth vs Ideal LP


Butterworth - 2/3

Gaussian LP Filtering



High Pass Filtering

- HP = 1 - LP
  - All the same filters as HP apply
- Applications
  - Visualization of high-freq data (accentuate)
- High boost filtering
  - HB = (1- a) + a(1 - LP) = 1 - a*LP

High-Pass Filters

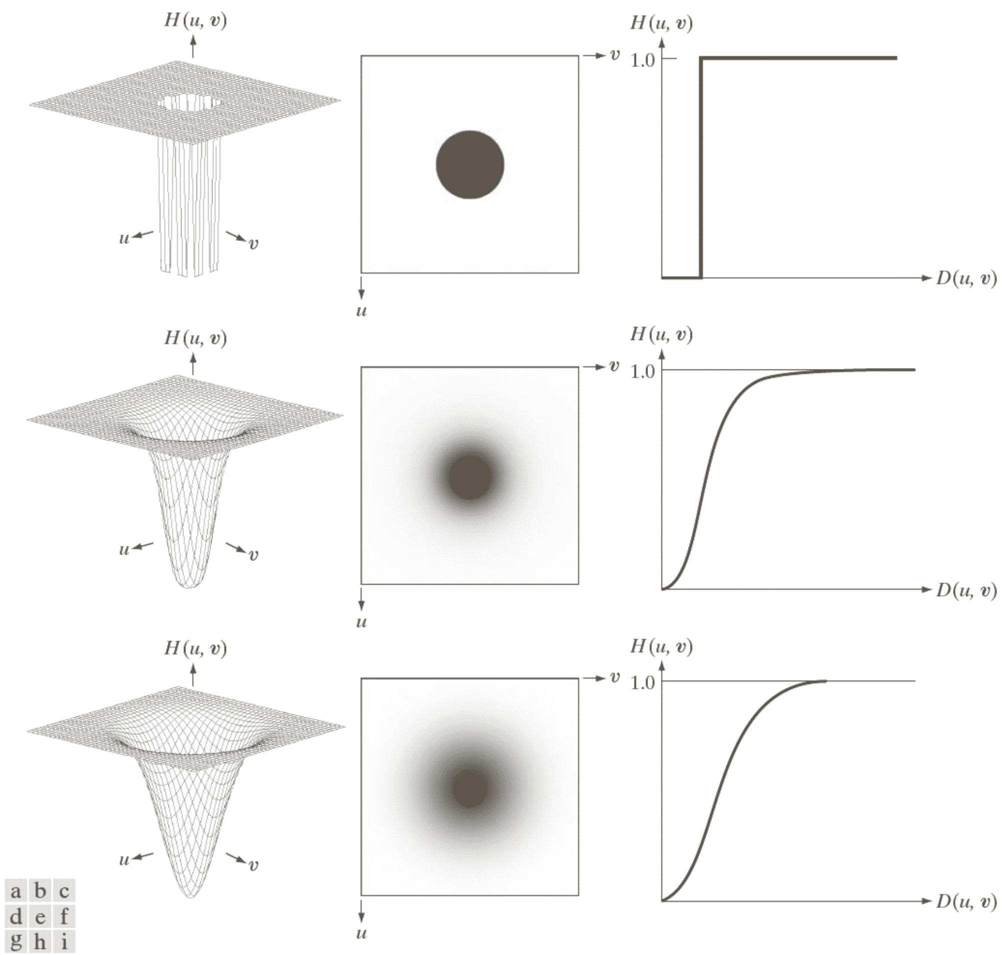
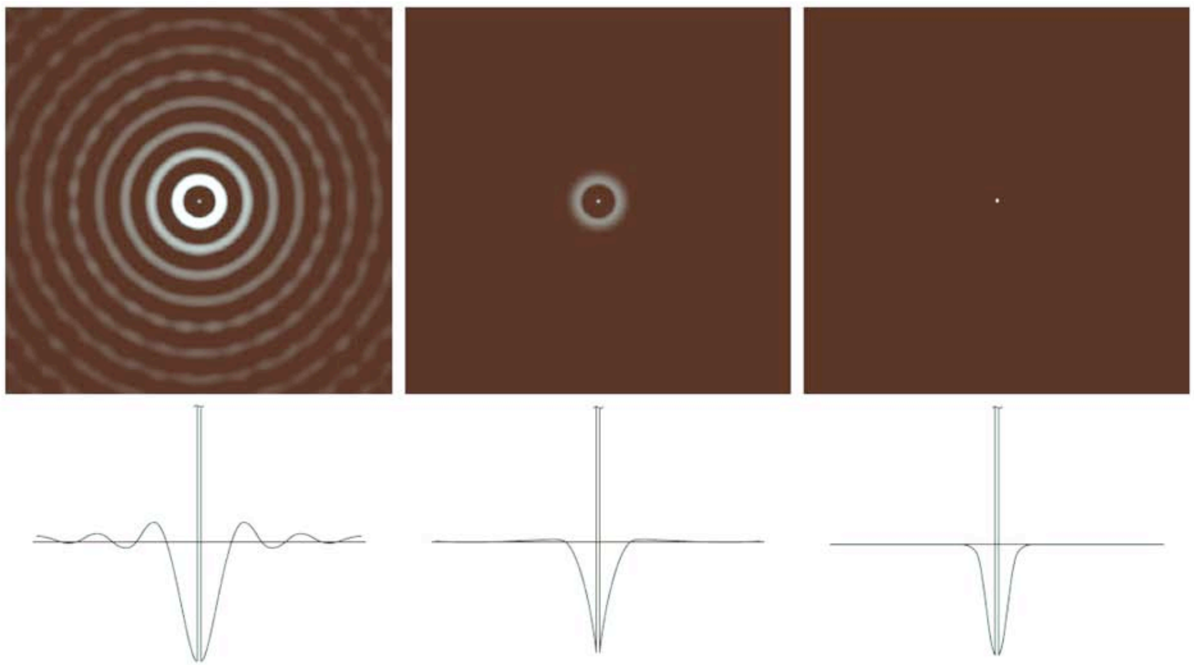


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

High-Pass Filters in Spatial Domain



a b c

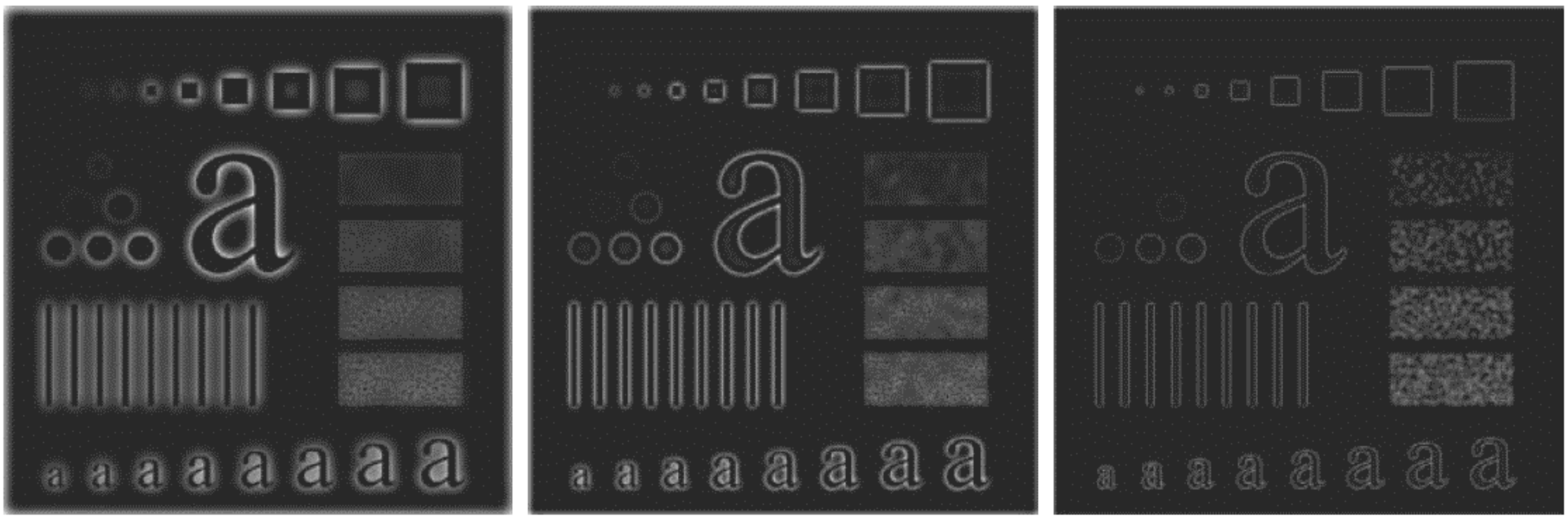
FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

High-Pass Filtering with IHPF

a b c

FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with $D_0 = 30$, 60, and 160.

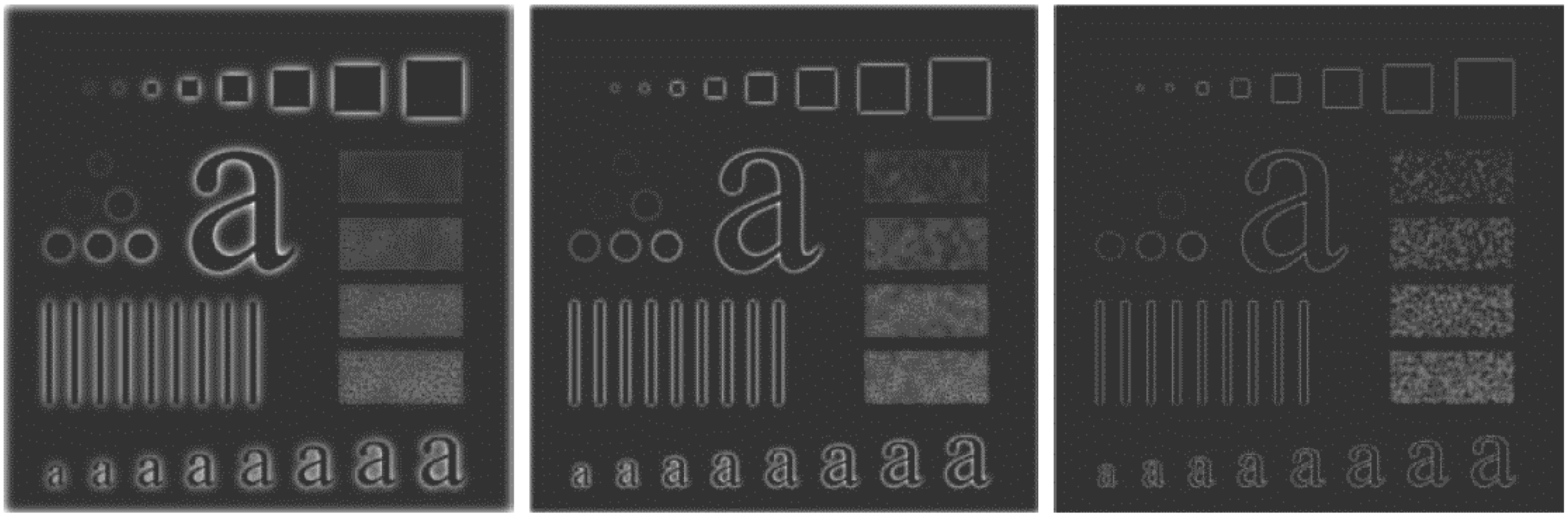
BHPF



a b c

FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

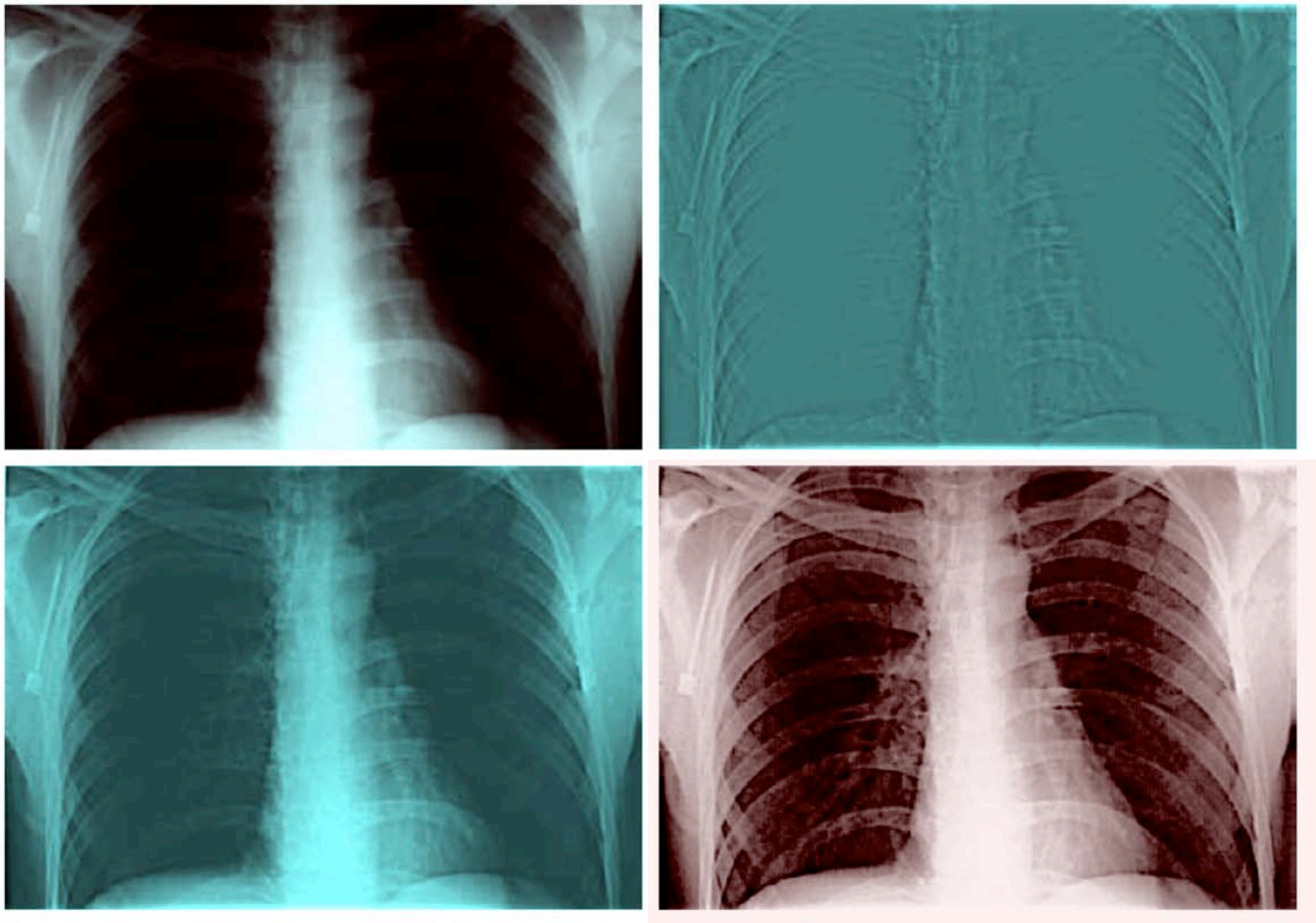
GHPF



a b c

FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with $D_0 = 30, 60,$ and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

HP, HB, HE



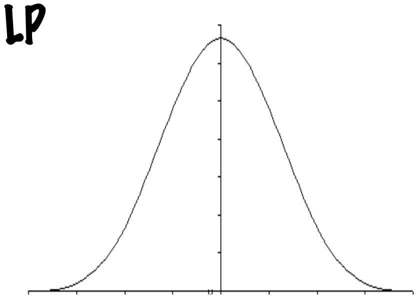
High Boost with GLPF

High-Boost Filtering

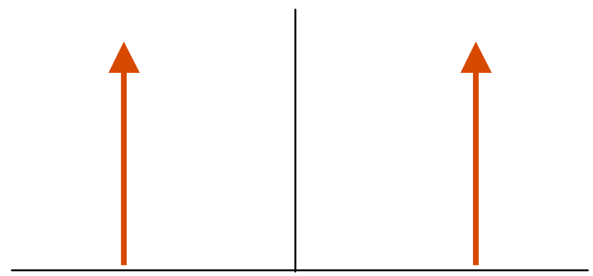
Band-Pass Filters

- Shift LP filter in Fourier domain by convolution with delta

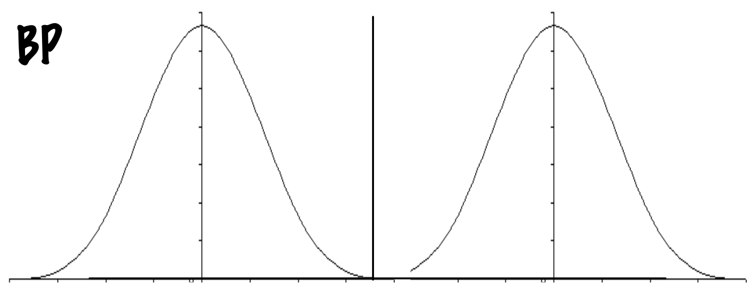
LP



$$\delta(s - s_0) + \delta(s + s_0)$$



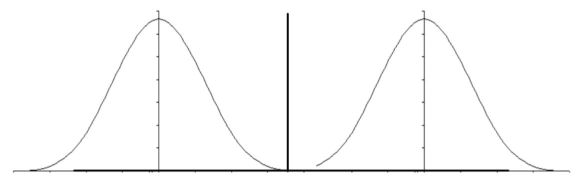
BP

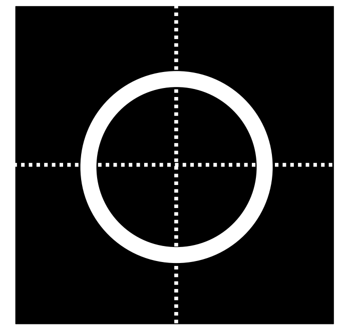


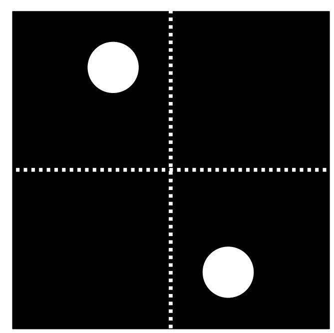
Typically 2-3 parameters
- -Width
- -Slope
- -Band value

Band Pass - Two Dimensions

- Two strategies
  - Rotate
    - Radially symmetric
  - Translate in 2D
    - Oriented filters

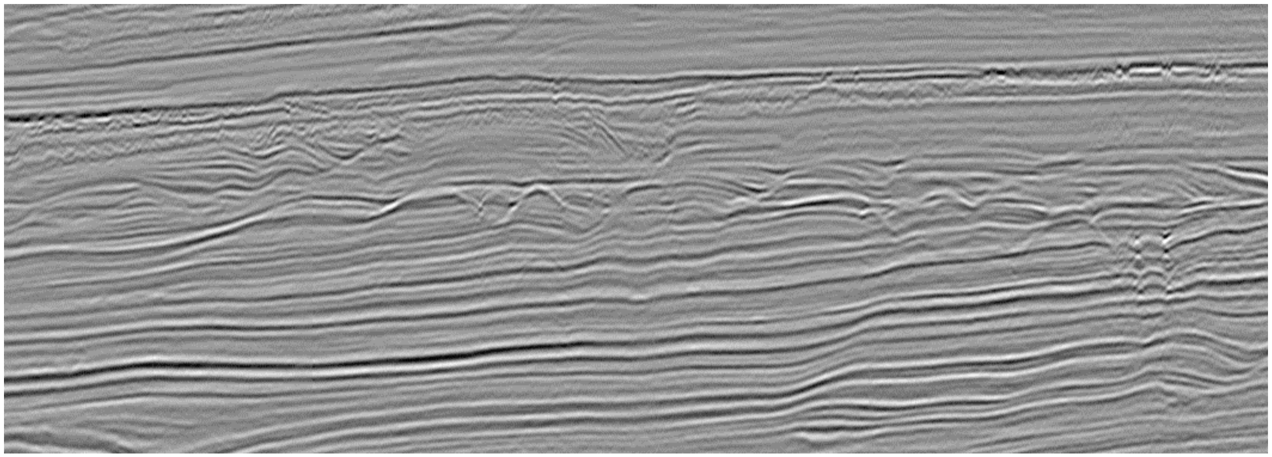


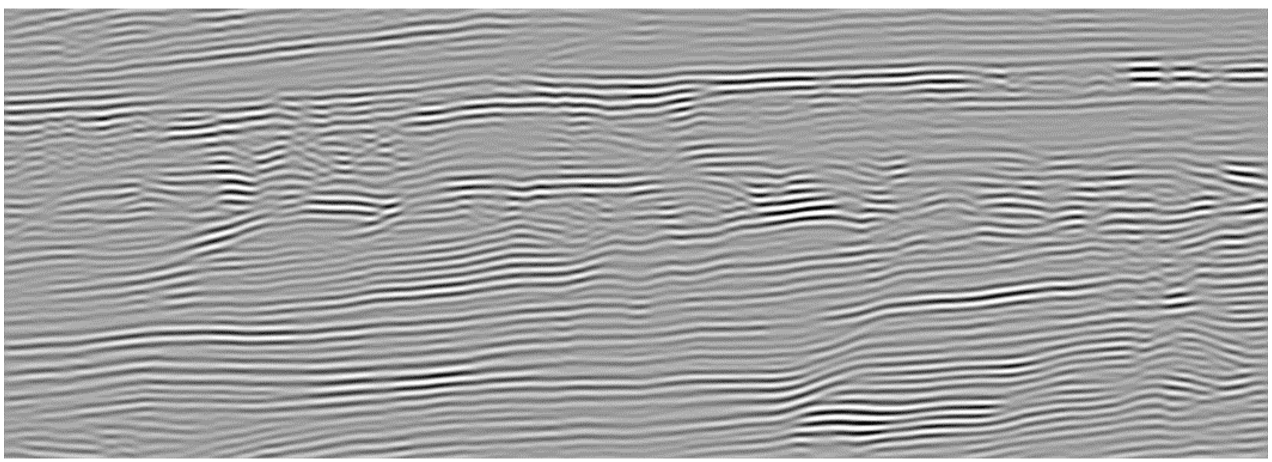




- Note:
  - Convolution with delta-pair in FD is multiplication with cosine in spatial domain

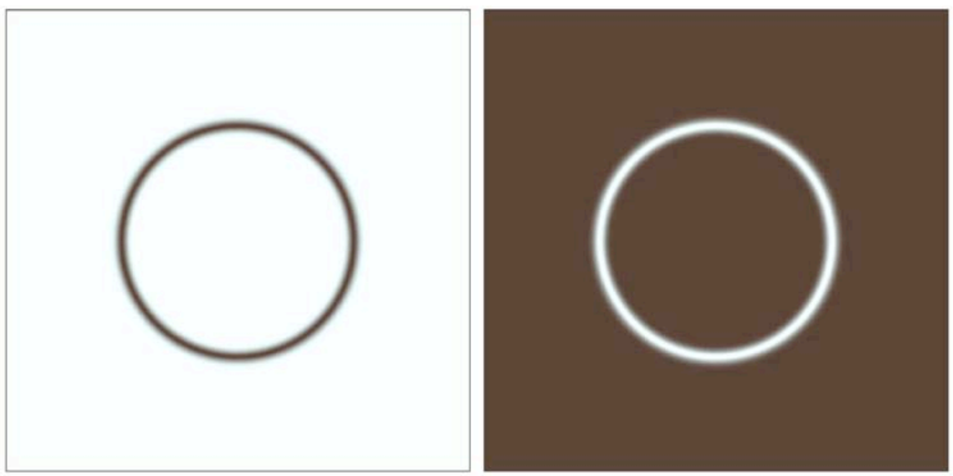
Band Bass Filtering



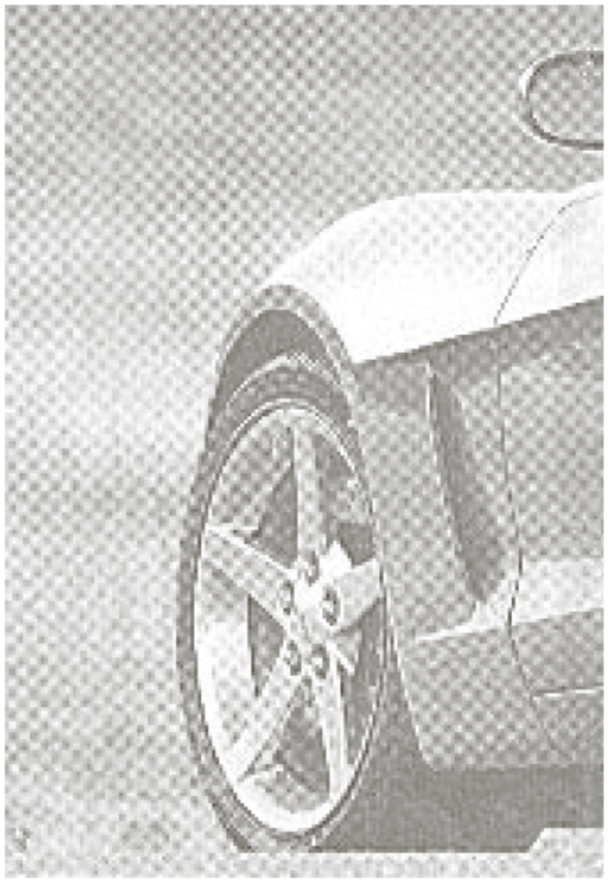


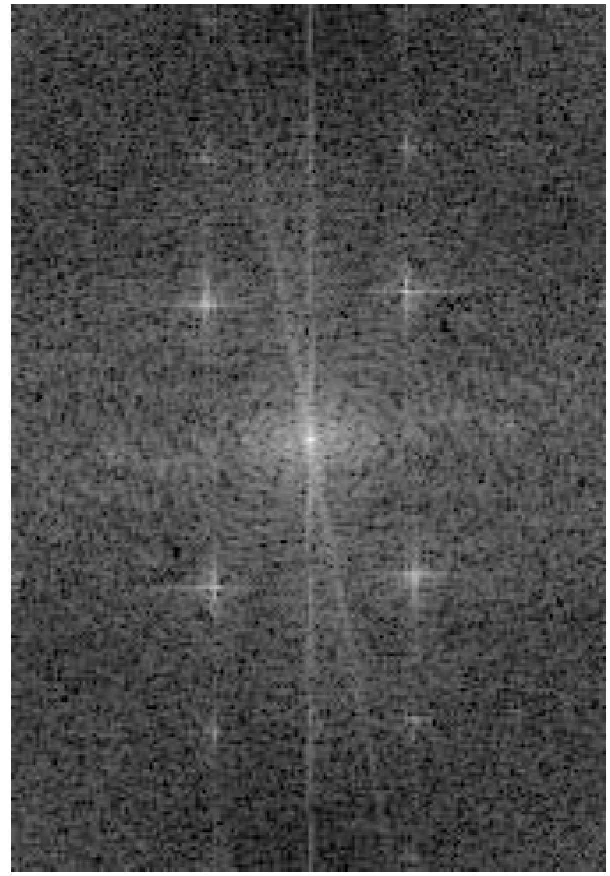
Radial Band Pass/Reject

Ideal	Butterworth	Gaussian
$H(u,v) = \begin{cases} 0 & \text{if } D_0 - \frac{W}{2} \leq D \leq D_0 + \frac{W}{2} \\ 1 & \text{otherwise} \end{cases}$	$H(u,v) = \dfrac{1}{1 + \left[\dfrac{DW}{D^2 - D_0^2}\right]^{2n}}$	$H(u,v) = 1 - e^{-\left[\frac{D^2 - D_0^2}{DW}\right]^2}$

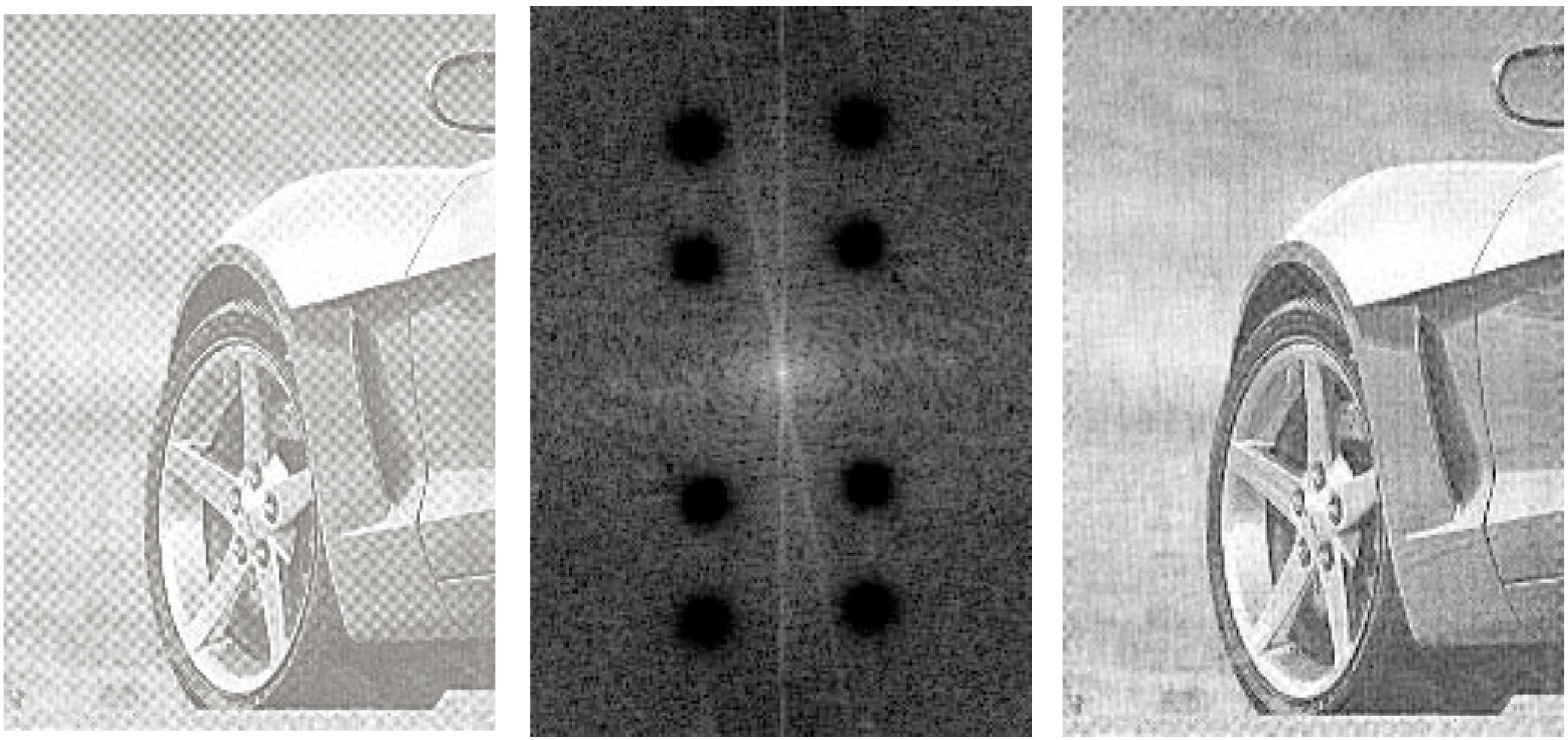


Band Reject Filtering

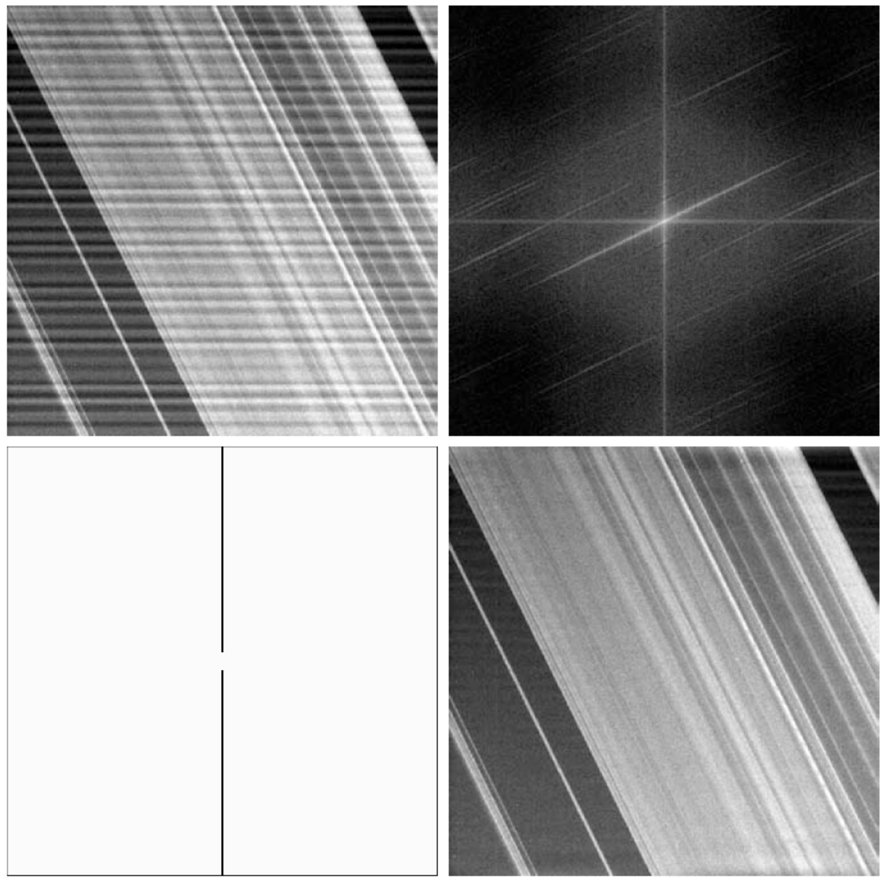




Band Reject Filtering



Band Reject Filtering

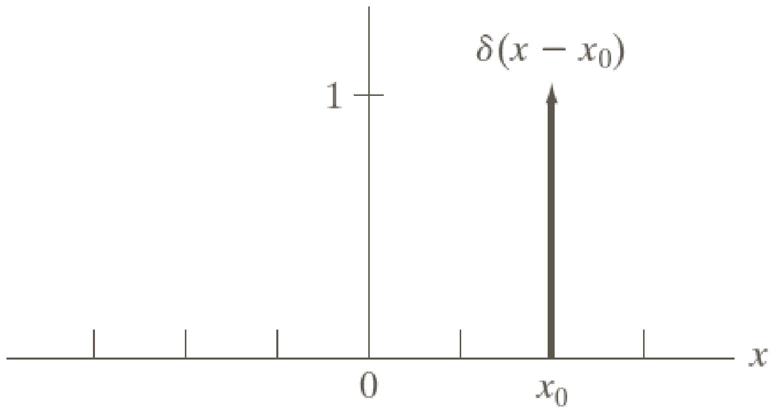


Discrete Sampling and Aliasing

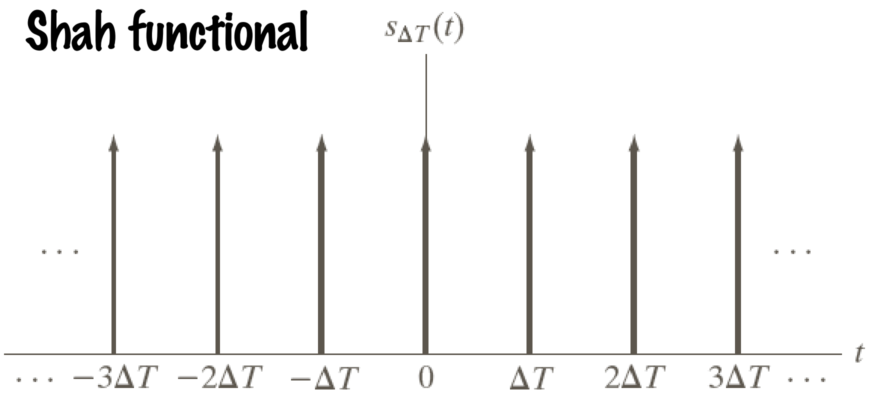
- Digital signals and images are discrete representations of the real world
  - Which is continuous
- What happens to signals/images when we sample them?
  - Can we quantify the effects?
  - Can we understand the artifacts and can we limit them?
  - Can we reconstruct the original image from the discrete data?

A Mathematical Model of Discrete Samples

Delta functional



Shah functional



$$s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

A Mathematical Model of Discrete Samples

- **Goal**
  - To be able to do a continuous Fourier transform on a signal before and after sampling

Discrete signal

$$f_k \quad k = 0, \pm 1, \ldots$$

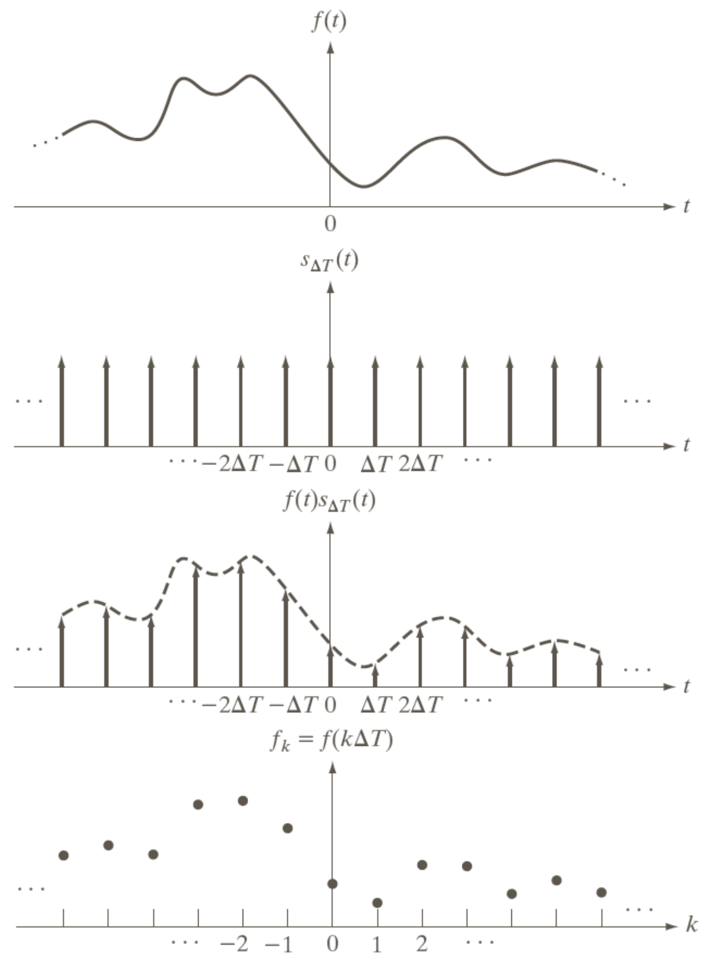
Samples from continuous function

$$f_k = f(k\Delta T)$$

Representation as a function of t

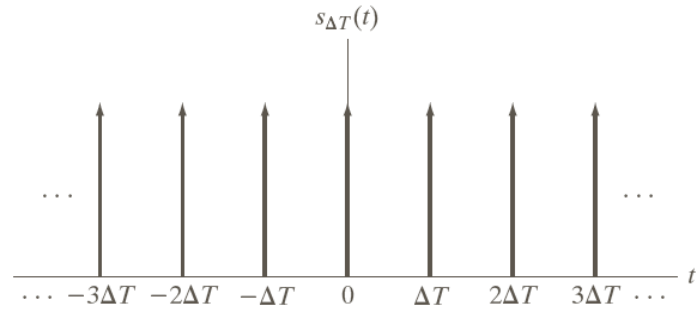
- Multiplication of f(t) with Shah

$$\tilde{f}(t) = f(t)s_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - k\Delta T)$$

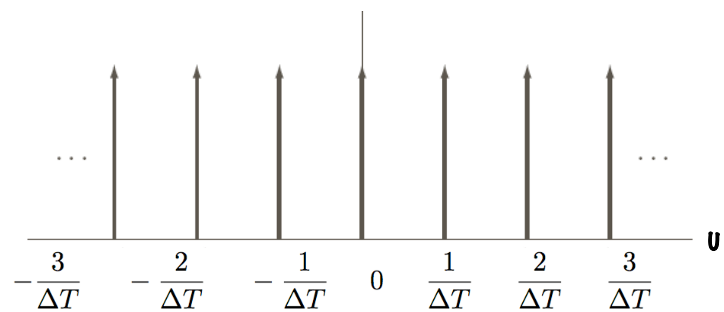


Fourier Series of A Shah Functional

$$s(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T)$$

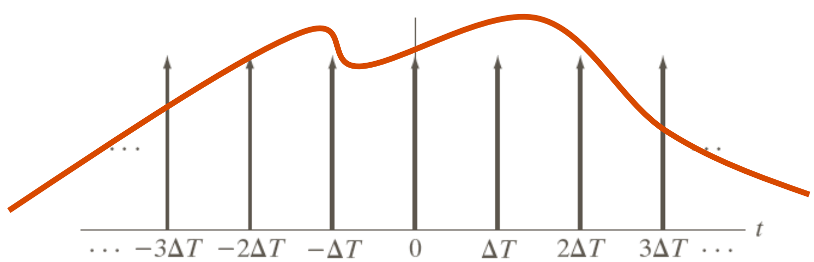


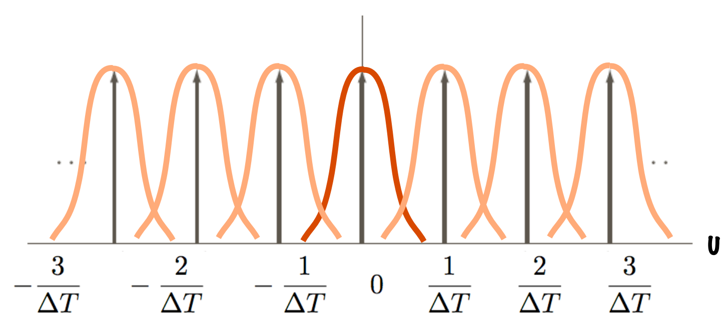
$$S(u) = \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} \delta(t - \frac{k}{\Delta T})$$



Fourier Transform of A Discrete Sampling

$$\tilde{f}(t) = f(t)s(t) \longleftrightarrow \tilde{F}(u) = F(u) * S(u)$$

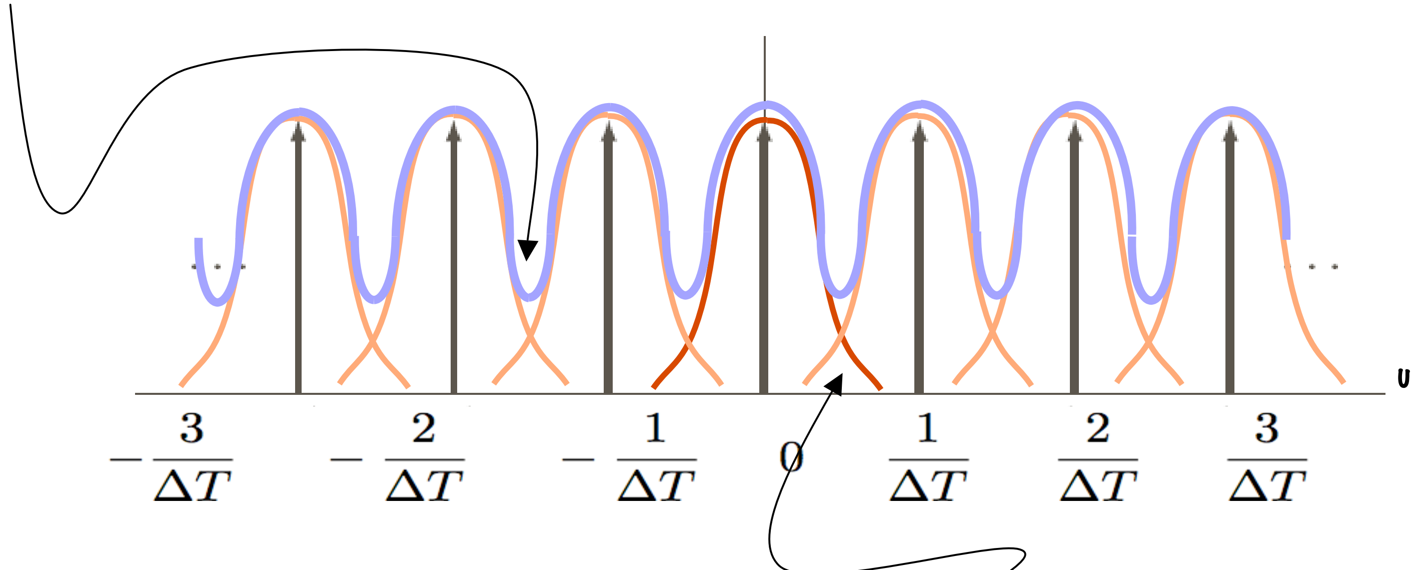




Fourier Transform of A Discrete Sampling

Frequencies get mixed. The original signal is not recoverable.

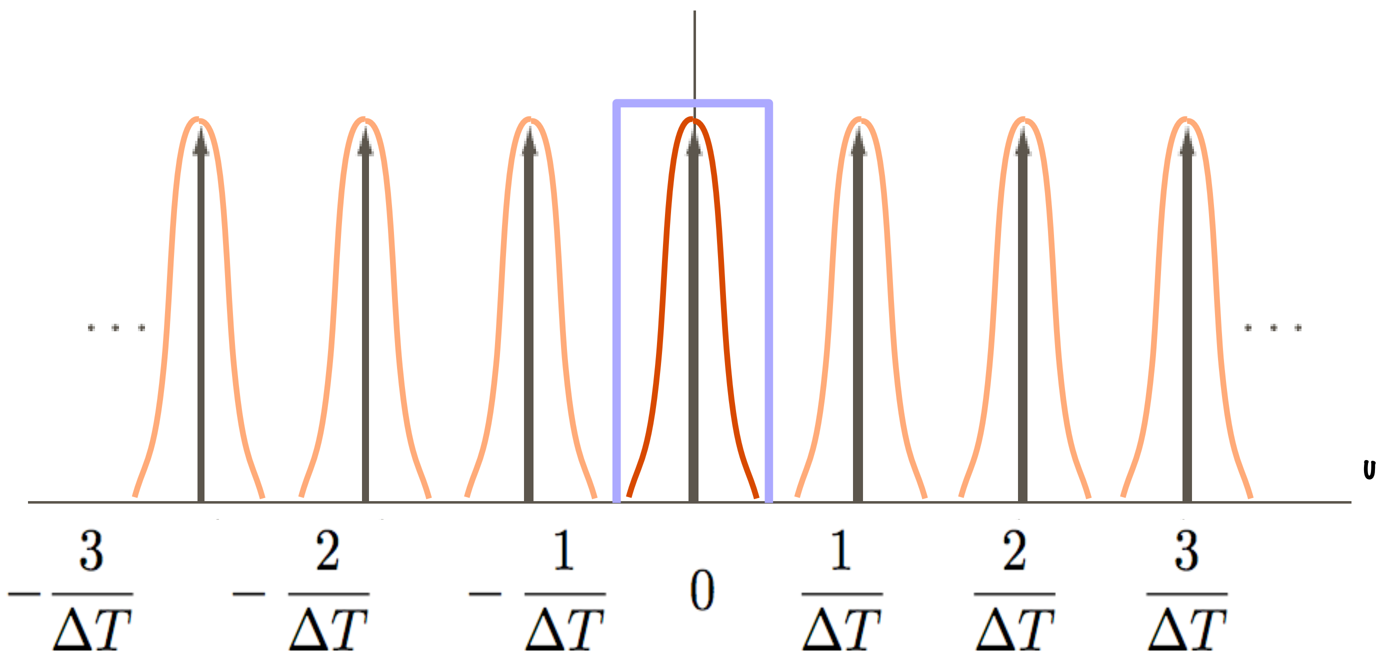
$$\tilde{F}(u) = F(u) * S(u)$$



Energy from higher freqs gets folded back down into lower freqs – Aliasing

What if F(u) is Narrower in the Fourier Domain?

- No aliasing!
- How could we recover the original signal?



What Comes Out of This Model

- Sampling criterion for complete recovery
- An understanding of the effects of sampling
  - Aliasing and how to avoid it
- Reconstruction of signals from discrete samples

Shannon Sampling Theorem

- Assuming a signal that is band limited:

$$f(t) \longleftrightarrow F(u) \qquad |F(u)| = 0 \ \forall \ |u| > B$$

- Given set of samples from that signal

$$f_k = f(k\Delta T) \qquad \Delta T \leq \frac{1}{2B}$$

- Samples can be used to generate the original signal
  - Samples and continuous signal are equivalent

Sampling Theorem

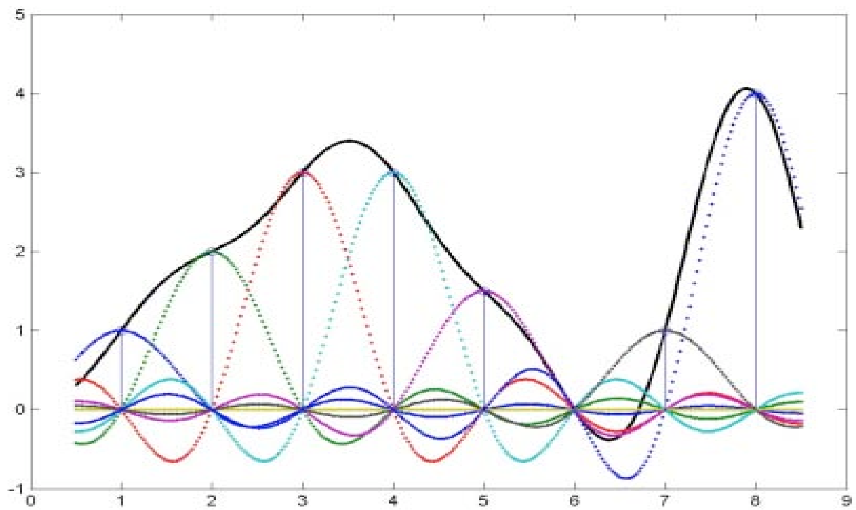
- Quantifies the amount of information in a signal
  - Discrete signal contains limited frequencies
  - Band-limited signals contain no more information then their discrete equivalents
- Reconstruction by cutting away the repeated signals in the Fourier domain
  - Convolution with sinc function in space/time

Reconstruction

- Convolution with sinc function

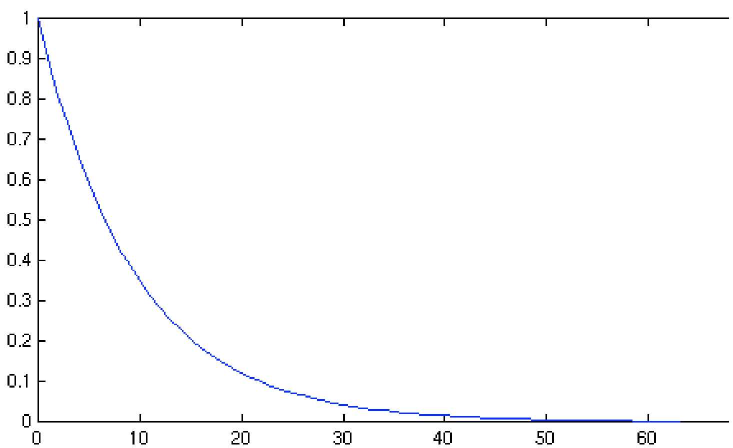
$$f(t) = \tilde{f}(t) * \mathbb{F}^{-1}\left[\mathrm{rect}\left(\frac{\mathrm{u}}{\Delta\mathrm{T}}\right)\right]$$

$$= \left(\sum_k f_k \delta(t - k\Delta T)\right) * \mathrm{sinc}\left(\frac{\mathrm{t}}{\Delta\mathrm{T}}\right) = \sum_{\mathrm{k}} \mathrm{f_k}\ \mathrm{sinc}\left(\frac{\mathrm{t} - \mathrm{k}\Delta\mathrm{T}}{\Delta\mathrm{T}}\right)$$

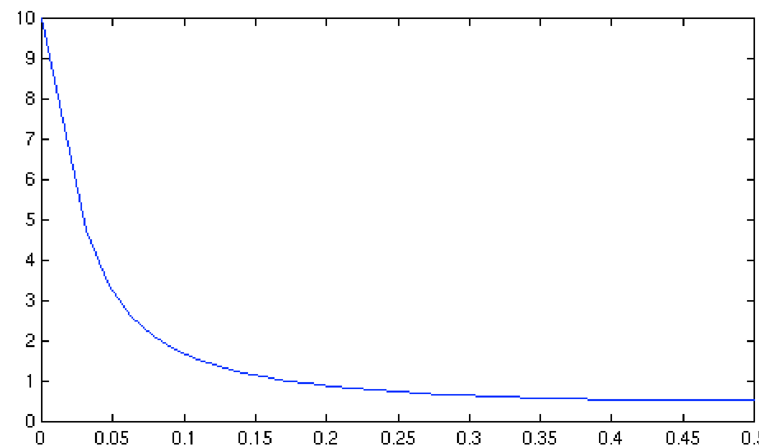


Sinc Interpolation Issues

- Must functions are not band limited
- Forcing functions to be band-limited can cause artifacts (ringing)

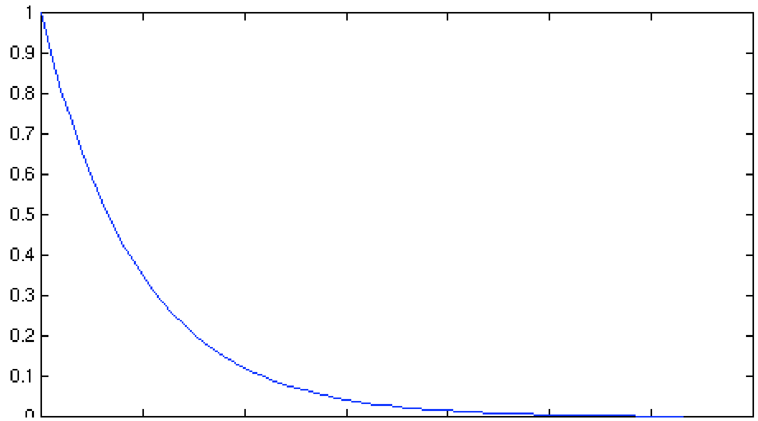


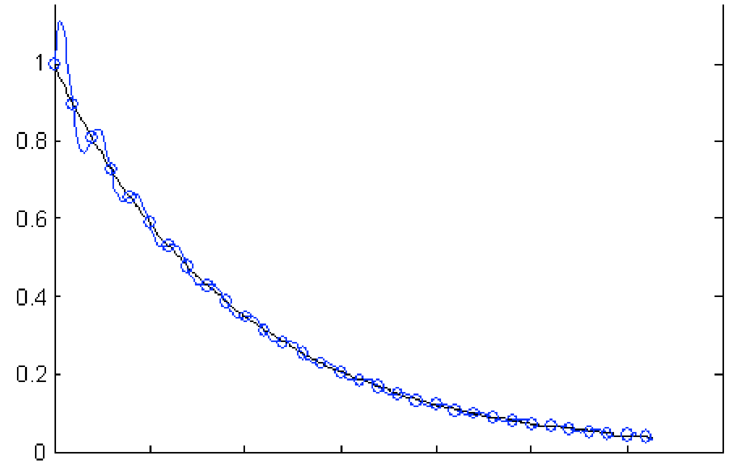
f(t)



|F(s)|

Sinc Interpolation Issues





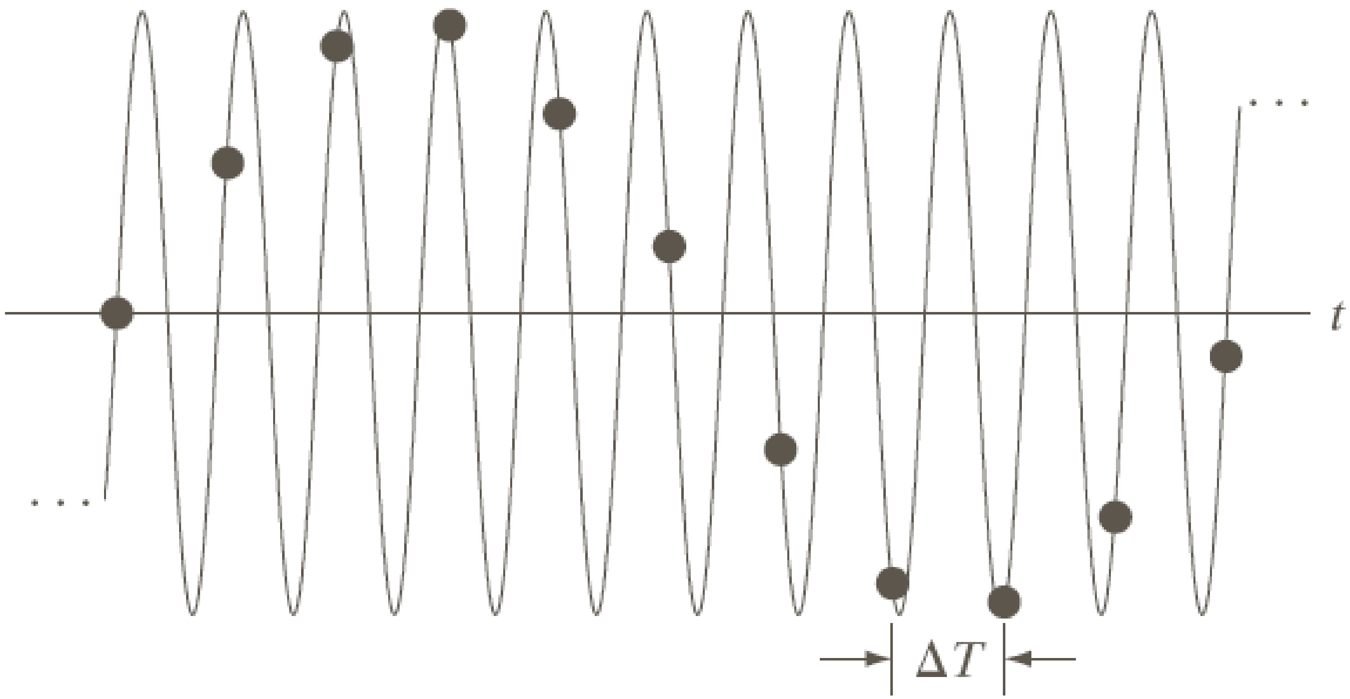
Ringing - Gibbs phenomenon

Other issues:

Sinc is infinite - must be truncated

Aliasing

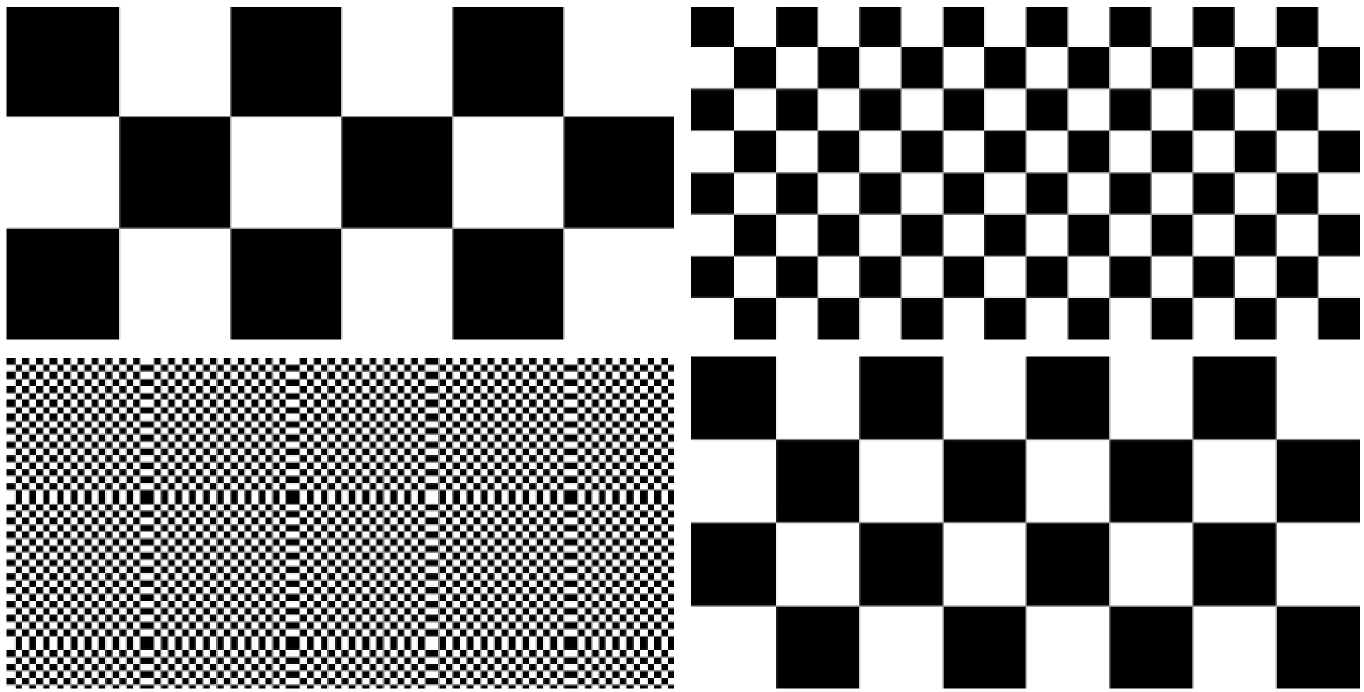
- High frequencies appear as low frequencies when undersampled



Aliasing

16 pixels

8 pixels

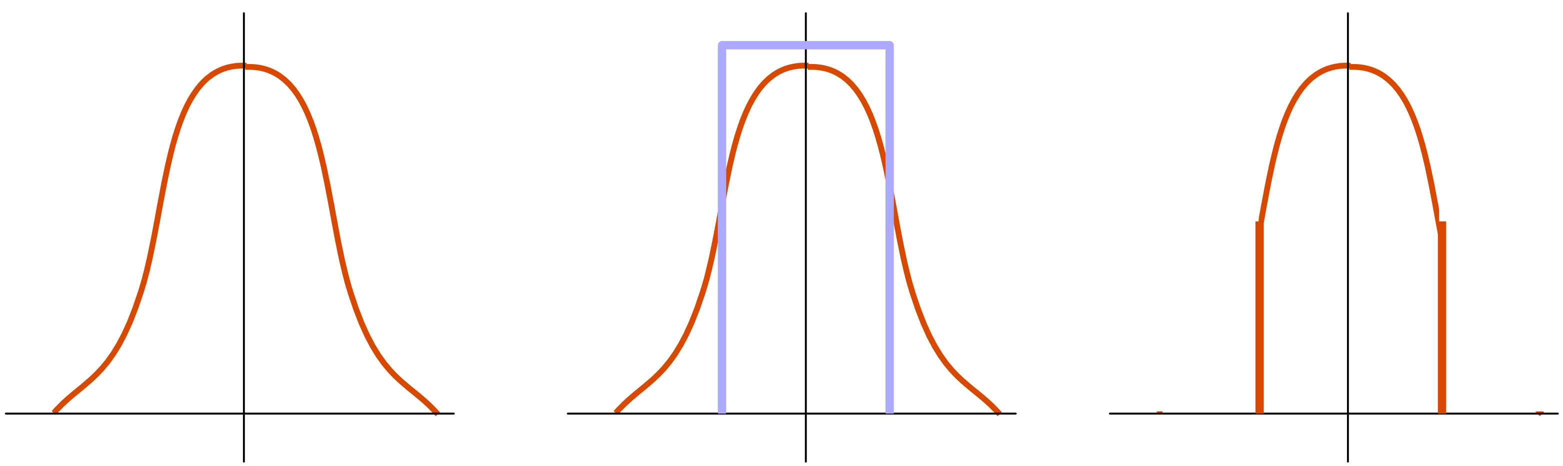


0.9174 pixels

0.4798 pixels

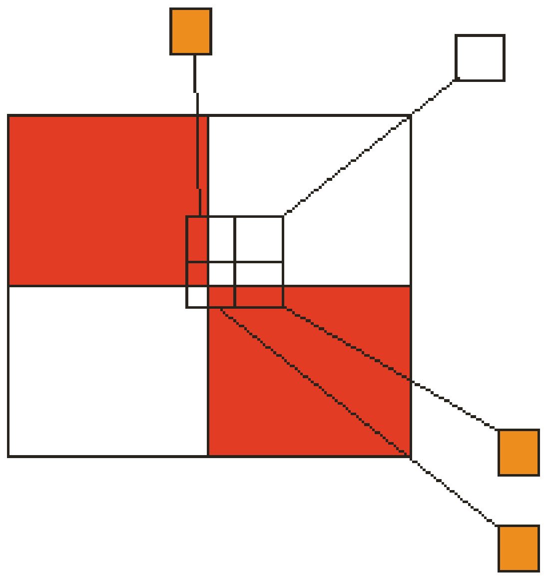
Overcoming Aliasing

- Filter data prior to sampling
  - Ideally - band limit the data (conv with sinc function)
  - In practice - limit effects with fuzzy/soft low pass



Antialiasing in Graphics

- Screen resolution produces aliasing on underlying geometry



Multiple high-res samples get averaged to create one screen sample

Antialiasing

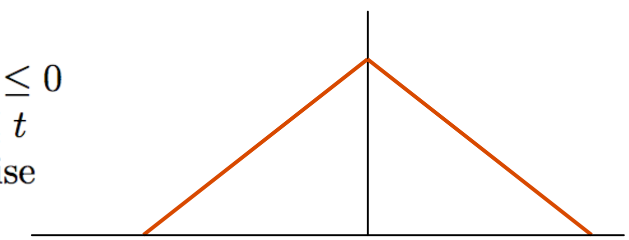
Interpolation as Convolution

- Any discrete set of samples can be considered as a functional

$$\tilde{f}(t) = \sum_k f_k \delta(t - k\Delta T)$$

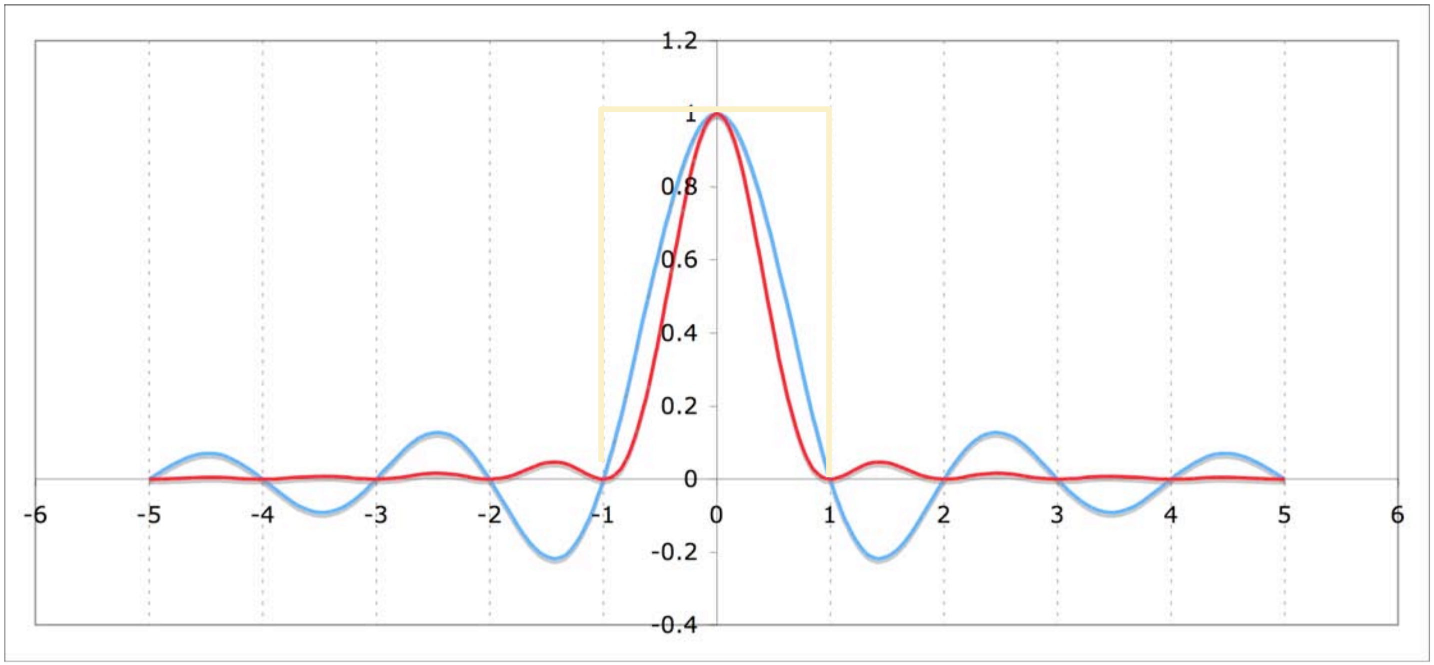
- Any linear interpolant can be considered as a convolution
  - Nearest neighbor - rect(t)
  - Linear - tri(t)

$$\mathrm{tri(t)} = \begin{cases} t+1 & -1 \le t \le 0 \\ 1-t & 0 \le t \le t \\ 0 & \text{otherwise} \end{cases}$$

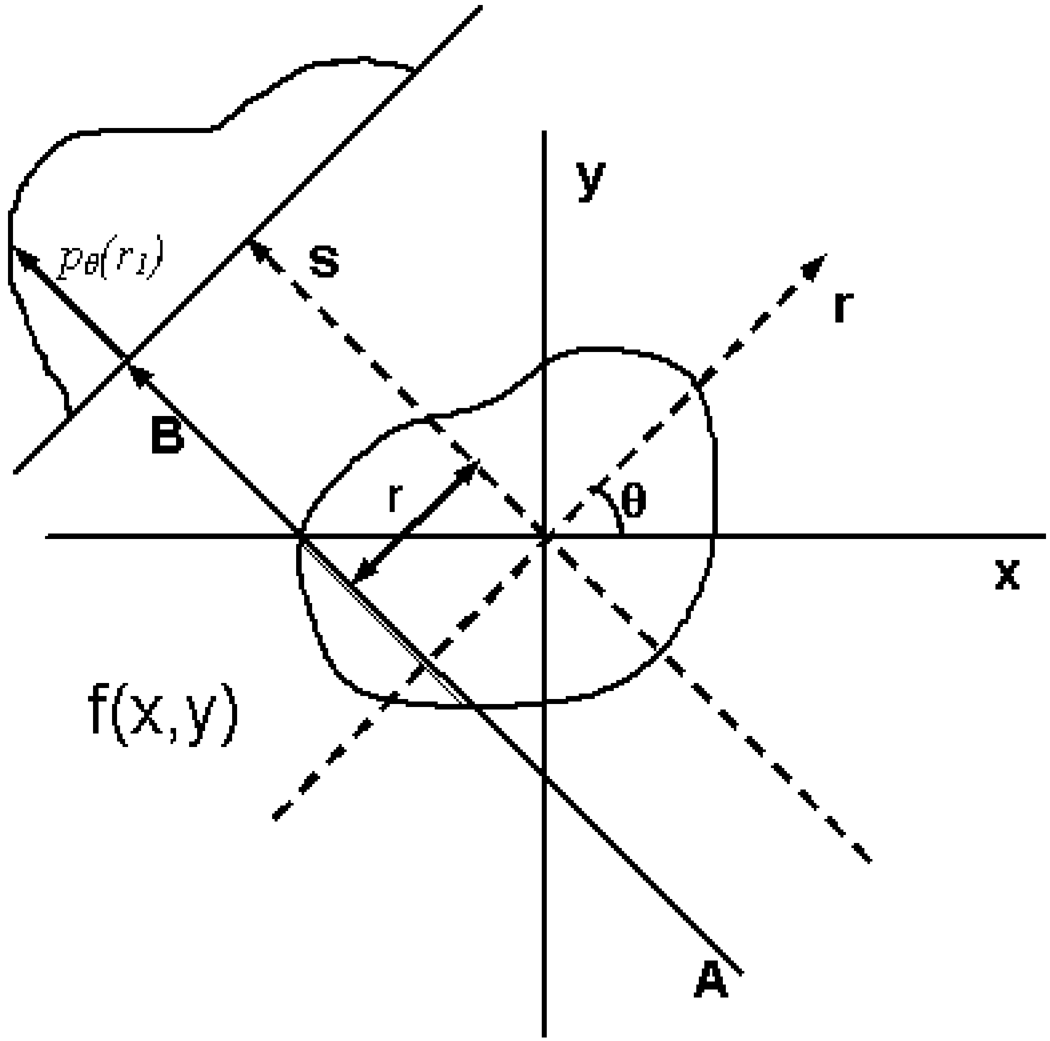


Convolution-Based Interpolation

- Can be studied in terms of Fourier Domain
- Issues
  - Pass energy (=1) in band
  - Low energy out of band
  - Reduce hard cut off (Gibbs, ringing)



Tomography



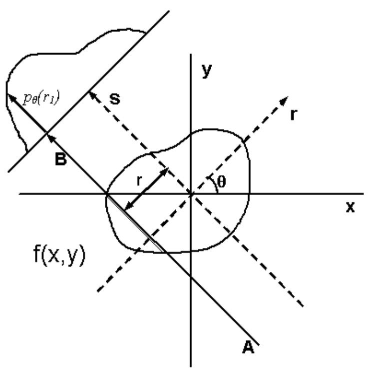
Tomography Formulation

Attenuation $$I = I_0 \exp\left(-\int \mu(x,y)\,ds\right)$$

Log gives line integral $$p(r,\theta) = \ln(I/I_0) = -\int \mu(x,y)\,ds$$

Line with angle theta $$x\cos\theta + y\sin\theta = r$$

Volume integral $$p(r,\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta + y\sin\theta - r)\,dx\,dy$$



Fourier Slice Theorem

$$F_1 P_1 = S_1 F_2$$

1D FT — F_1; Projection to 1D — P_1; 1D Slice — S_1; 2D FT — F_2