Geometric Transformations and Image Warping

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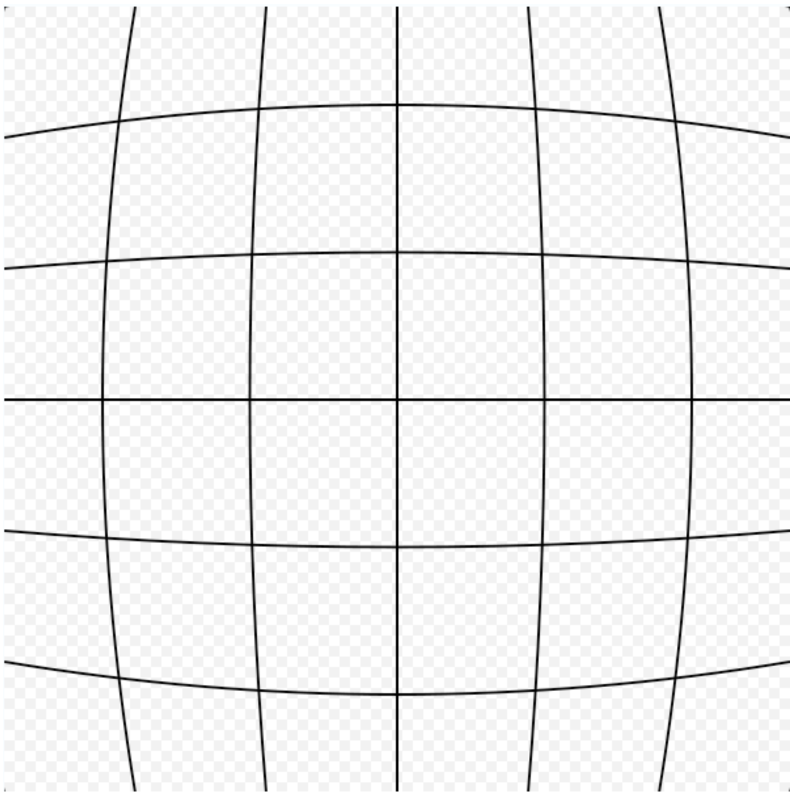
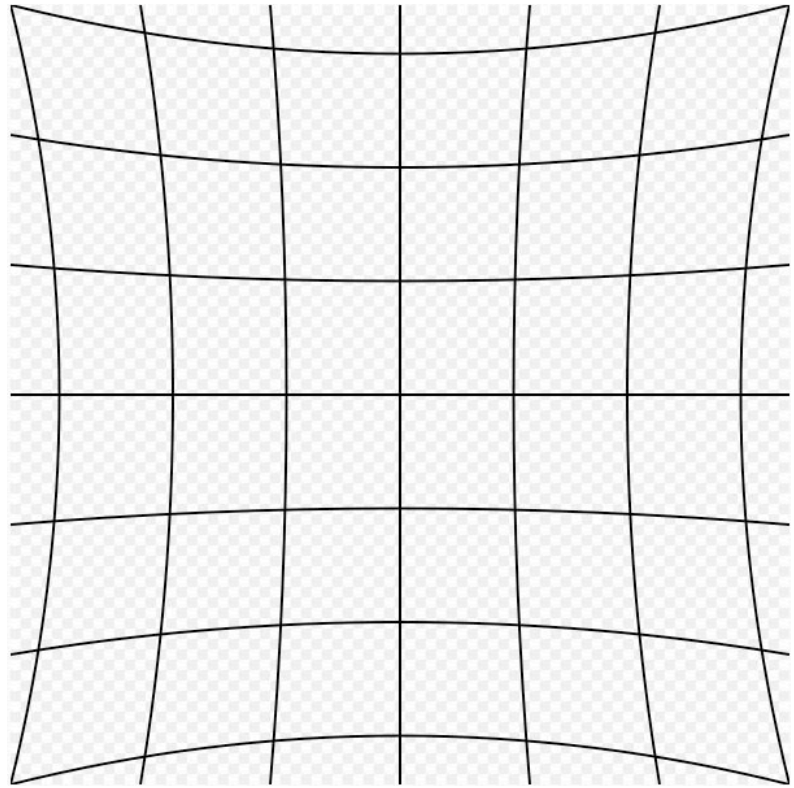
Geometric Transformations

- Greyscale transformations -> operate on range/ output
- Geometric transformations -> operate on image domain
  - Coordinate transformations
  - Moving image content from one place to another
- Two parts:
  - Define transformation
  - Resample greyscale image in new coordinates

Geom Trans: Distortion From Optics



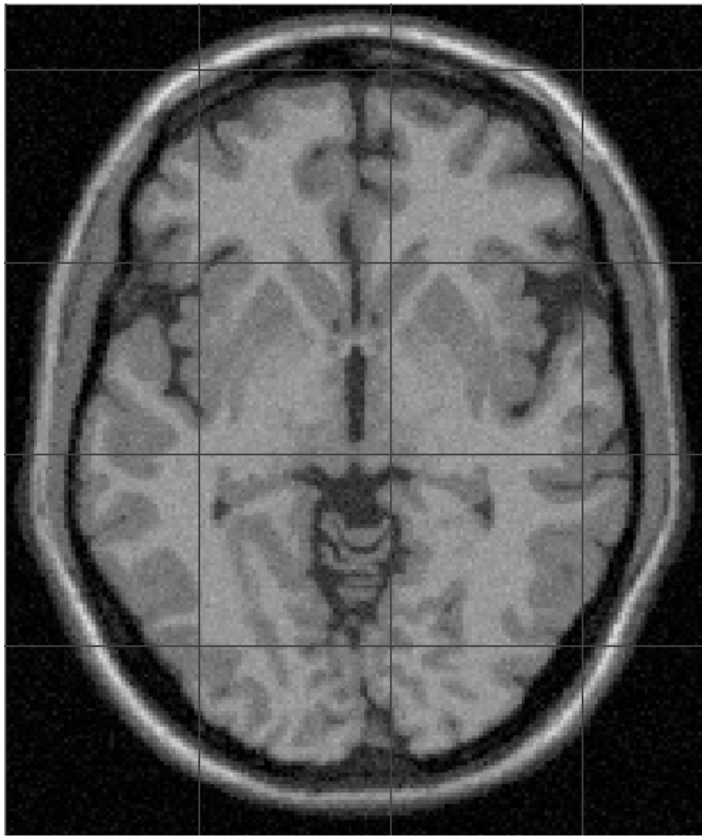
Barrel Distortion

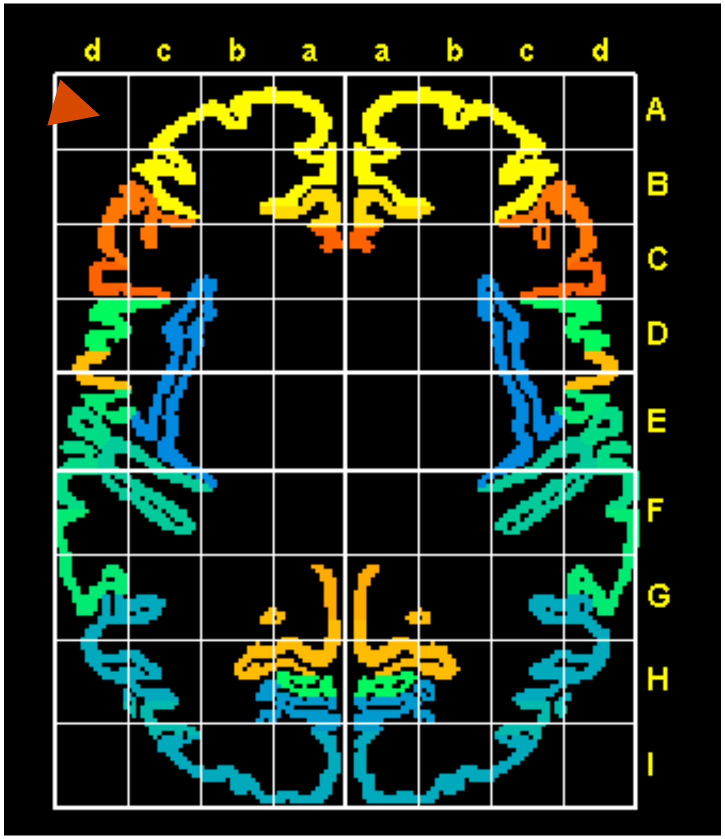


Pincushion Distortion

Geom Trans: Distortion From Optics

Geom. Trans.: Brain Template/ Atlas





Geom. Trans.: Mosaicing

Domain Mappings Formulation

$$f \longrightarrow g$$

New image from old one

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = T(x, y) = \begin{pmatrix} T_1(x, y) \\ T_2(x, y) \end{pmatrix}$$

Coordinate transformation
Two parts - vector valued

$$g(x, y) = f(x', y')$$
$$g(x, y) = f(x', y') = \tilde{f}(x, y)$$

g is the same image as f, but sampled on these new coordinates

Domain Mappings Formulation

$$\bar{x}' = T(\bar{x})$$

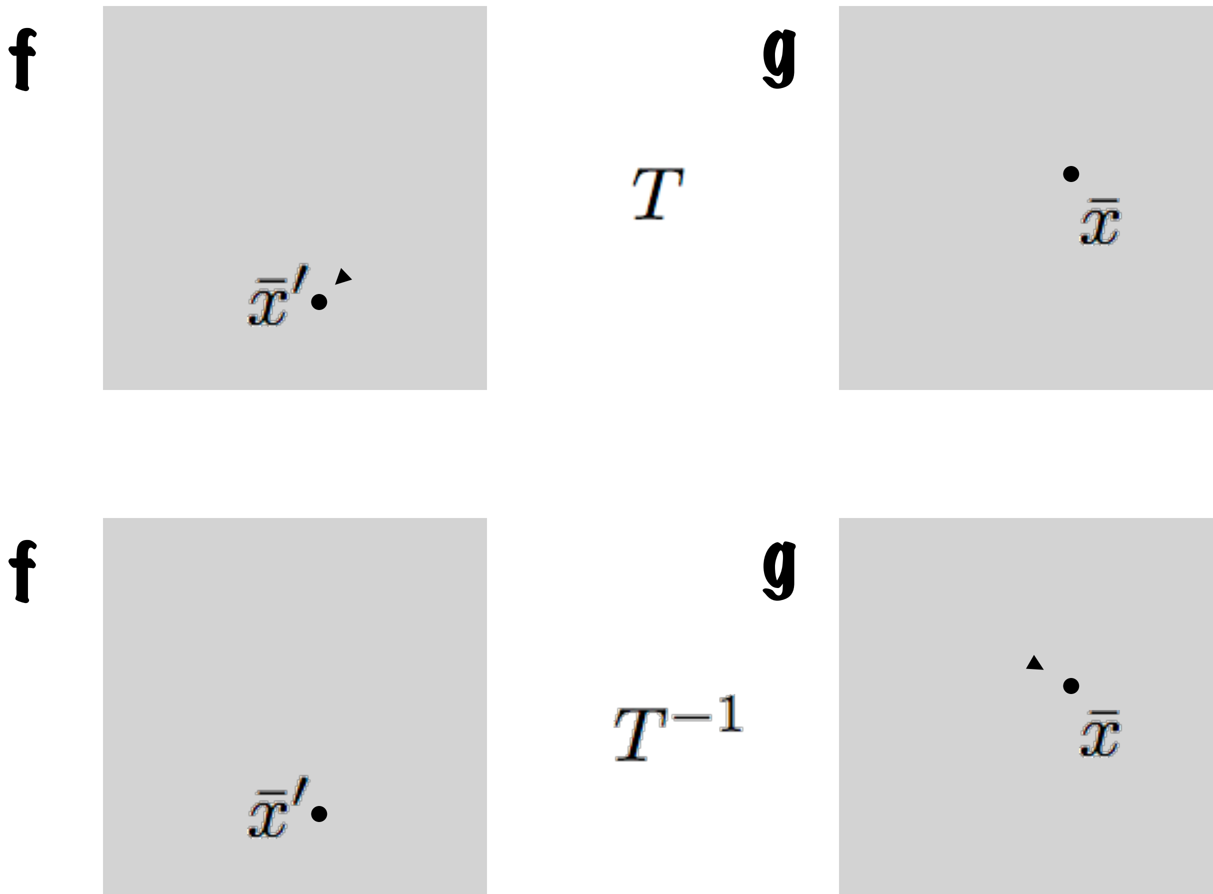
Vector notation is convenient. Bar used some times, depends on context.

$$g(\bar{x}) = \tilde{f}(\bar{x}) = f(\bar{x}') = f(T(\bar{x}))$$

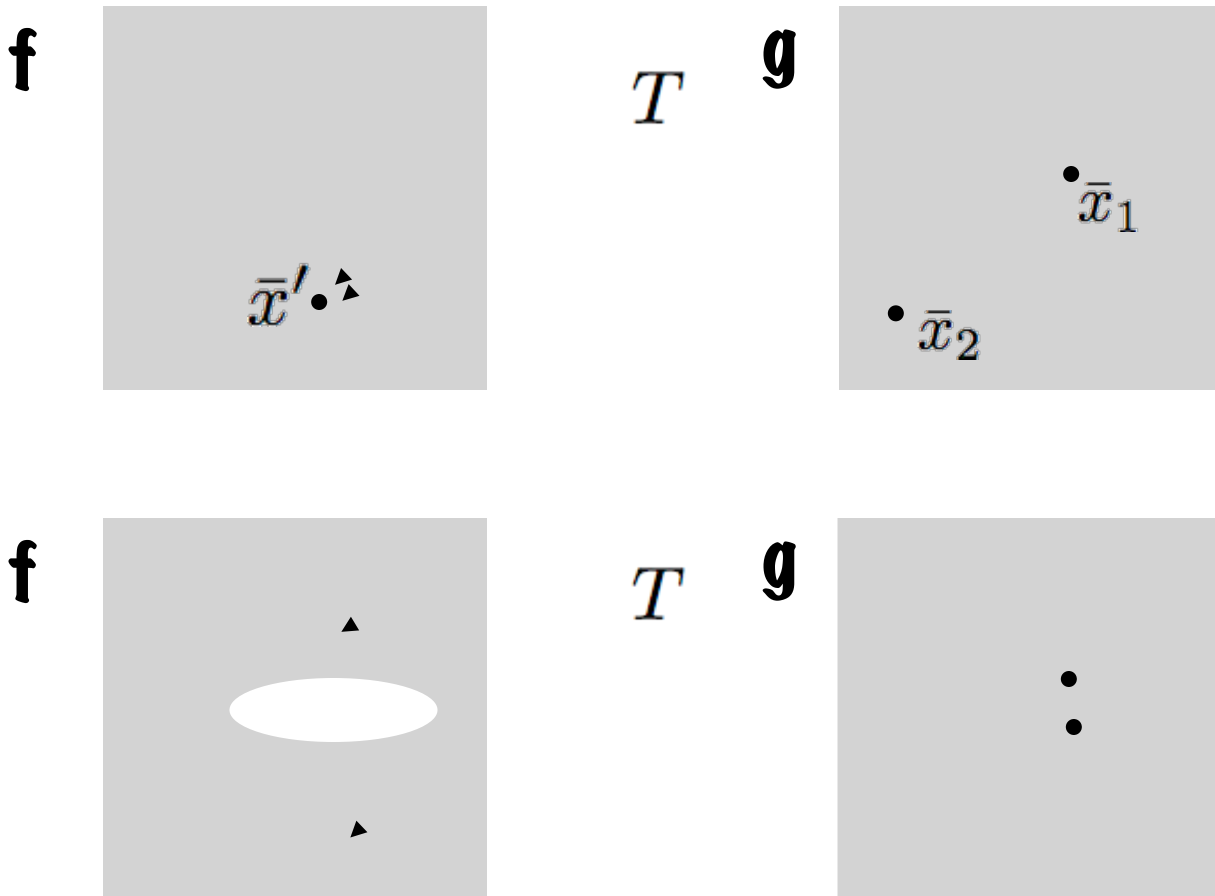
$$\bar{x} = T^{-1}(\bar{x}')$$

T may or may not have an inverse. If not, it means that information was lost.

Domain Mappings



No Inverse?

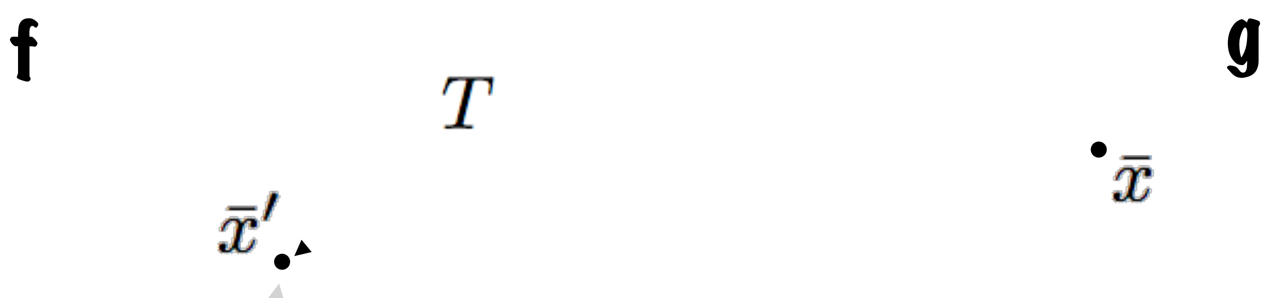


Not "one to one"

Not "onto" - doesn't cover f

Implementation - Two Approaches

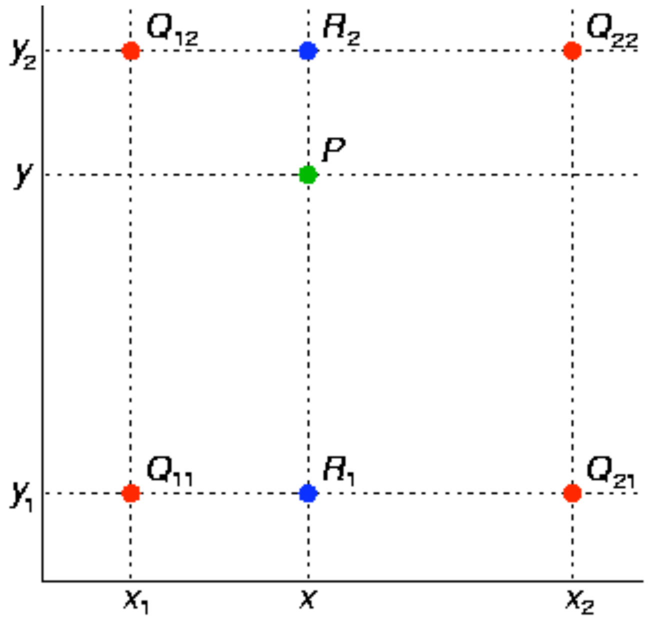
1. Pixel filling - backward mapping
   - T() takes you from coords in g() to coords in f()
   - Need random access to pixels in f()
   - Sample grid for g(), interpolate f() as needed



Interpolate from nearby grid points

Interpolation: Binlinear

- Successive application of linear interpolation along each axis



$$f(R_1) \approx \frac{x_2 - x}{x_2 - x_1} f(Q_{11}) + \frac{x - x_1}{x_2 - x_1} f(Q_{21})$$

$$f(R_2) \approx \frac{x_2 - x}{x_2 - x_1} f(Q_{12}) + \frac{x - x_1}{x_2 - x_1} f(Q_{22})$$

$$f(P) \approx \frac{y_2 - y}{y_2 - y_1} f(R_1) + \frac{y - y_1}{y_2 - y_1} f(R_2).$$

Source: WIkipedia

Binlinear Interpolation

- *Not* linear in x, y

$$f(x,y) \approx \frac{f(Q_{11})}{(x_2-x_1)(y_2-y_1)}(x_2-x)(y_2-y) + \frac{f(Q_{21})}{(x_2-x_1)(y_2-y_1)}(x-x_1)(y_2-y) + \frac{f(Q_{12})}{(x_2-x_1)(y_2-y_1)}(x_2-x)(y-y_1) + \frac{f(Q_{22})}{(x_2-x_1)(y_2-y_1)}(x-x_1)(y-y_1).$$

$$b_1 + b_2 x + b_3 y + b_4 xy$$

$$\begin{aligned} b_1 &= f(0,0) \\ b_2 &= f(1,0) - f(0,0) \\ b_3 &= f(0,1) - f(0,0) \\ b_4 &= f(0,0) - f(1,0) \\ &\quad - f(0,1) + f(1,1). \end{aligned}$$

Binlinear Interpolation

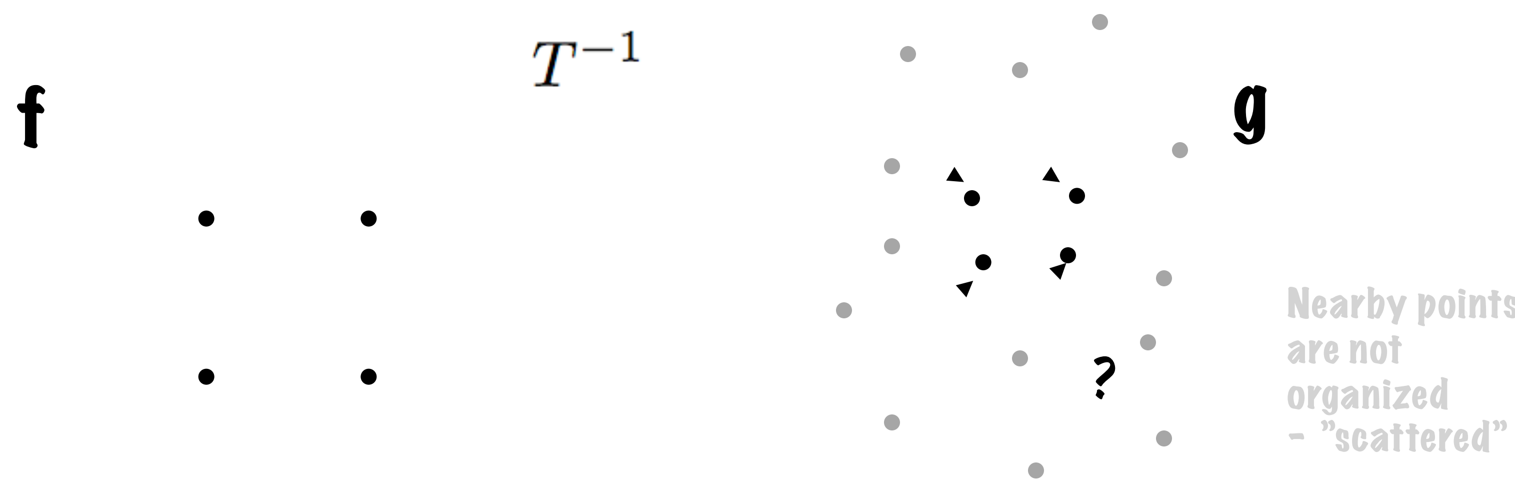
- Convenient form
  - Normalize to unit grid [0,1]x[0,1]

$$f(x,y) \approx f(0,0)\,(1-x)(1-y) + f(1,0)\,x(1-y) + f(0,1)\,(1-x)y + f(1,1)xy.$$

$$f(x,y) \approx \begin{bmatrix} 1-x & x \end{bmatrix} \begin{bmatrix} f(0,0) & f(0,1) \\ f(1,0) & f(1,1) \end{bmatrix} \begin{bmatrix} 1-y \\ y \end{bmatrix}.$$

Implementation - Two Approaches

2. Splatting - backward mapping
   - T^{-1}() takes you from coords in f() to coords in g()
   - You have f() on grid, but you need g() on grid
   - Push grid samples onto g() grid and do interpolation from unorganized data (kernel)



Scattered Data Interpolation With Kernels Shepard's method

- **Define kernel**
  - Falls off with distance, radially symmetric

Kernel examples

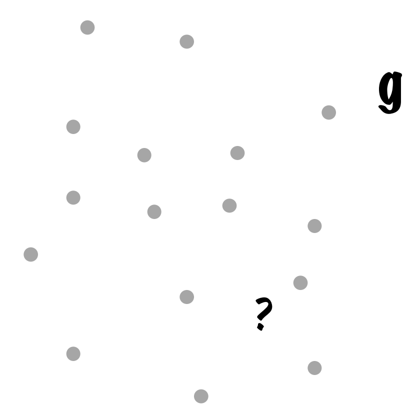
$$K(\bar{x}_1, \bar{x}_2) = \frac{1}{2\pi\sigma^2} e^{\frac{|\bar{x}_1 - \bar{x}_2|^2}{2\sigma^2}}$$

$$K(\bar{x}_1, \bar{x}_2) = \frac{1}{|\bar{x}_1 - \bar{x}_2|^p}$$

$$K(\bar{x}_1, \bar{x}_2) = K(|\bar{x}_1 - \bar{x}_2|)$$

$$g(x) = \frac{1}{\sum_{j=1}^{N} w_j} \sum_{i=1}^{N} w_i f(x_i')$$

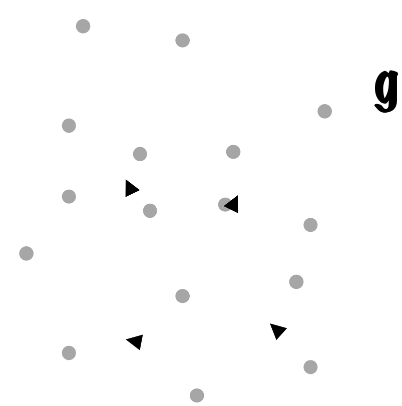
$$w_j = K\left(|\bar{x} - T^{-1}(\bar{x}_j')\right)$$



Shepard's Method Implementation

- If points are dense enough
  - Truncate kernel
  - For each point in f()
    - Form a small box around it in g() – beyond which truncate
    - Put weights and data onto grid in g()
  - Divide total data by total weights: B/A

Data and weights accumulated here



$$A = \sum_{j=1}^{N} w_j \qquad B = \sum_{i=1}^{N} w_i f\left(T^{-1}(x_i')\right)$$

Transformation Examples

- **Linear** $\bar{x}' = A\bar{x} + \bar{x}_0$ $\quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$x' = ax + by + x_0$$
$$y' = cx + dy + y_0$$

- **Homogeneous coordinates**

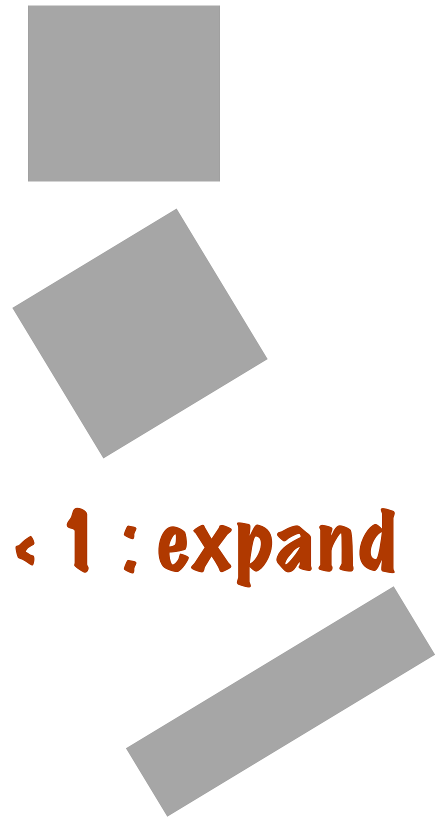
$$\bar{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \quad A = \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{x}' = A\bar{x}$$

Special Cases of Linear

- **Translation** $$A = \begin{pmatrix} 0 & 0 & x_0 \\ 0 & 0 & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$
- **Rotation** $$A = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
- **Rigid = rotation + translation**
- **Scaling** $$A = \begin{pmatrix} p & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  $p, q < 1$: expand
  - Include forward and backward rotation for arbitary axis



- **Skew**

- **Reflection**

Linear Transformations

- Also called "affine"
  - 6 parameters
- Rigid -> 3 parameters
- Invertability
  - Invert matrix

$$T^{-1}(\bar{x}) = A^{-1}\bar{x}$$

- What does it mean if A is not invertible?

Other Transformations

- All polynomials of (x,y)
- Any vector valued function with 2 inputs
- How to construct transformations
  - Define form or class of a transformation
  - Choose parameters within that class
    - Rigid - 3 parameters
    - Affine - 6 parameters

Correspondences

- Also called "landmarks" or "fiducials"

$$\bar{c}_1\,,\,\bar{c}_1'$$
$$\bar{c}_2\,,\,\bar{c}_2'$$
$$\bar{c}_3\,,\,\bar{c}_3'$$
$$\bar{c}_4\,,\,\bar{c}_4'$$
$$\bar{c}_5\,,\,\bar{c}_5'$$
$$\bar{c}_6\,,\,\bar{c}_6'$$

Transformations/Control Points Strategy

1. Define a functional representation for T with k parameters (B) $T(\beta, \bar{x})$ $\beta = (\beta_1, \beta_2, \ldots, \beta_K)$
2. Define (pick) N correspondences
3. Find B so that

$$\bar{c}_i' = T(\beta, \bar{c}_i) \quad i = 1, \ldots, N$$

4. If overconstrained (K < 2N) then solve

$$\arg\min_{\beta} \left[\sum_{\mathrm{i}=1}^{\mathrm{N}} \left(\bar{\mathrm{c}}_{\mathrm{i}}' - \mathrm{T}(\beta, \bar{\mathrm{c}}_{\mathrm{i}}) \right)^2 \right]$$

Example: Quadratic

Transformation

$$T_x = \beta_x^{00} + \beta_x^{10} x + \beta_x^{01} y + \beta_x^{11} xy + \beta_x^{20} x^2 + \beta_x^{02} y^2$$
$$T_y = \beta_y^{00} + \beta_y^{10} x + \beta_y^{01} y + \beta_y^{11} xy + \beta_y^{20} x^2 + \beta_y^{02} y^2$$

Denote $\bar{c}_i = (c_{x,i}, c_{y,i})$

Correspondences must match

$$c'_{y,i} = \beta_y^{00} + \beta_y^{10} c_{x,i} + \beta_y^{01} c_{y,i} + \beta_y^{11} c_{x,i} c_{y,i} + \beta_y^{20} c_{x,i}^2 + \beta_y^{02} c_{y,i}^2$$
$$c'_{x,i} = \beta_x^{00} + \beta_x^{10} c_{x,i} + \beta_x^{01} c_{y,i} + \beta_x^{11} c_{x,i} c_{y,i} + \beta_x^{20} c_{x,i}^2 + \beta_x^{02} c_{y,i}^2$$

Note: these equations are linear in the unkowns

Write As Linear System

$$\begin{pmatrix} \begin{matrix} 1 & c_{x,1} & c_{y,1} & c_{x,1}c_{y,1} & c_{x,1}^2 & c_{y,1}^2 \\ 1 & c_{x,2} & c_{y,2} & c_{x,2}c_{y,2} & c_{x,2}^2 & c_{y,2}^2 \\ & & & \vdots & & \\ 1 & c_{x,N} & c_{y,N} & c_{x,N}c_{y,N} & c_{x,N}^2 & c_{y,N}^2 \end{matrix} & 0 \\ 0 & \begin{matrix} 1 & c_{x,1} & c_{y,1} & c_{x,1}c_{y,1} & c_{x,1}^2 & c_{y,1}^2 \\ 1 & c_{x,2} & c_{y,2} & c_{x,2}c_{y,2} & c_{x,2}^2 & c_{y,2}^2 \\ & & & \vdots & & \\ 1 & c_{x,N} & c_{y,N} & c_{x,N}c_{y,N} & c_{x,N}^2 & c_{y,N}^2 \end{matrix} \end{pmatrix} \begin{pmatrix} \beta_x^{00} \\ \beta_x^{10} \\ \beta_x^{01} \\ \beta_x^{11} \\ \beta_x^{20} \\ \beta_x^{02} \\ \beta_y^{00} \\ \beta_y^{10} \\ \beta_y^{01} \\ \beta_y^{11} \\ \beta_y^{20} \\ \beta_y^{02} \end{pmatrix} = \begin{pmatrix} c'_{x,1} \\ c'_{x,2} \\ \vdots \\ c'_{x,N} \\ c'_{y,1} \\ c'_{y,2} \\ \vdots \\ c'_{y,N} \end{pmatrix}$$

$$Ax = b$$

A - matrix that depends on the (unprimed) correspondences and the transformation

x - unknown parameters of the transformation

b - the primed correspondences

Linear Algebra Background

$$Ax = b$$

$$\begin{array}{rcl} a_{11}x_1 + \ldots + a_{1N}x_N & = & b_1 \\ a_{21}x_1 + \ldots + a_{2N}x_N & = & b_2 \\ \ldots & & \ldots \\ a_{M1}x_1 + \ldots + a_{MN}x_N & = & b_M \end{array}$$

Simple case: A is sqaure (M=N) and invertable (det[A] not zero)

$$A^{-1}Ax = Ix = x = A^{-1}b$$

Numerics: Don't find A inverse. Use Gaussian elimination or some kind of decomposition of A

Linear Systems - Other Cases

- M<N or M = N and the equations are degenerate or *singular*
  - System is underconstrained - lots of solutions
- Approach
  - Impose some extra criterion on the solution
  - Find the one solution that optimizes that criterion
  - *Regularizing* the problem

Linear Systems - Other Cases

- M > N
  - System is overconstrained
  - *No* solution
- Approach
  - Find solution that is best compromise
  - Minimize squared error (least squares)

$$x = \arg\min_{\mathrm{x}} |\mathrm{Ax} - \mathrm{b}|^2$$

Solving Least Squares Systems

- Psuedoinverse (normal equations)

$$A^T A x = A^T b$$
$$x = (A^T A)^{-1} A^T b$$

  - Issue: often not well conditioned (nearly singular)
- Alternative: *singular value decomposition*

Singular Value Decomposition

$$\begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} = UWV^T = \begin{pmatrix} & & \\ & U & \\ & & \end{pmatrix} \begin{pmatrix} w_1 & & & 0 \\ & w_2 & & \\ & & \cdots & \\ & & \cdots & \\ 0 & & & w_N \end{pmatrix} \begin{pmatrix} & V^T & \end{pmatrix}$$

$$I = U^T U = UU^T = V^T V = VV^T$$

Invert matrix A with SVD

$$A^{-1} = VW^{-1}U^T \qquad W^{-1} = \begin{pmatrix} \frac{1}{w_1} & & & 0 \\ & \frac{1}{w_2} & & \\ & & \cdots & \\ & & \cdots & \\ 0 & & & \frac{1}{w_N} \end{pmatrix}$$

SVD for Singular Systems

- If a system is singular, some of the w's will be zero

$$x = VW^*U^Tb$$

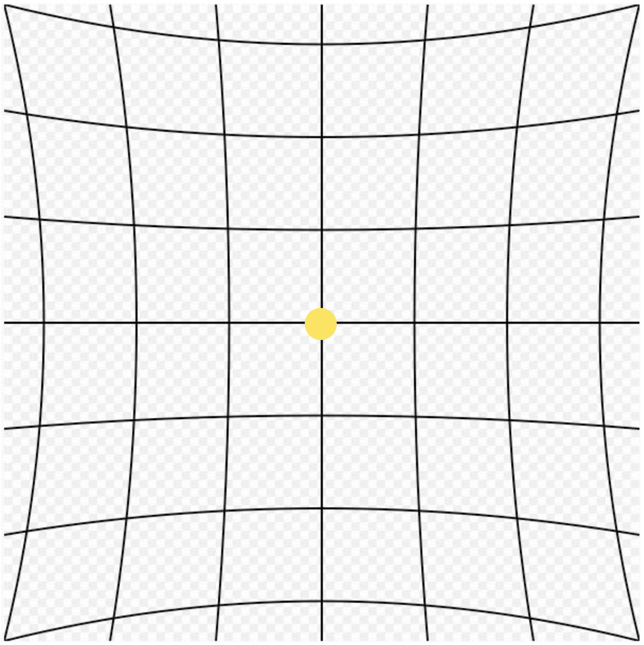
$$w_j^* = \begin{cases} 1/w_j & |w_j| > \epsilon \\ 0 & \text{otherwise} \end{cases}$$

- Properties:
  - Underconstrained: solution with shortest overall length
  - Overconstrained: least squares solution

Warping Application: Lens Distortion

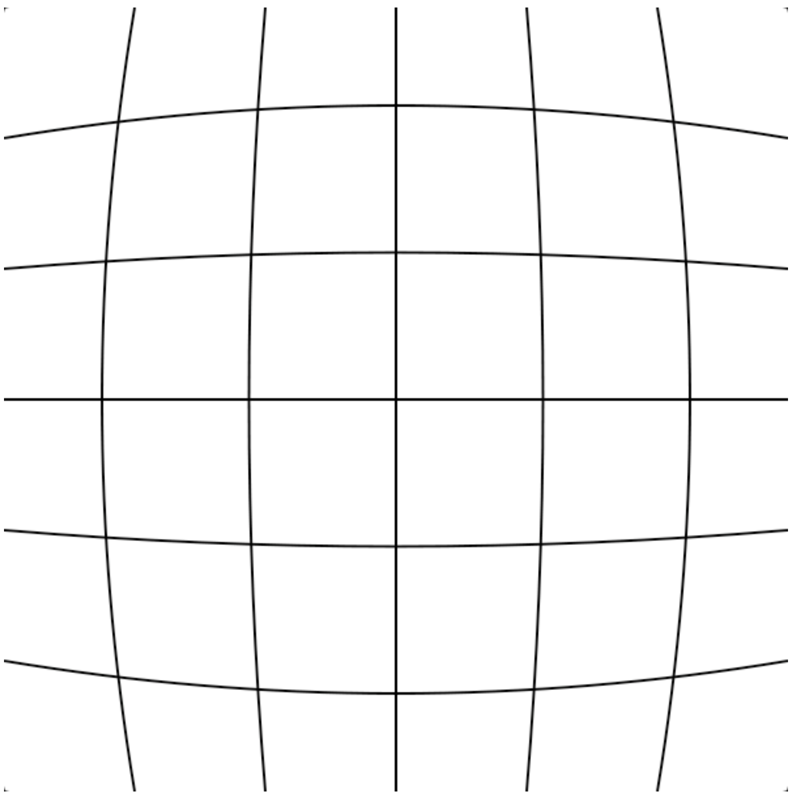
- Radial transformation – lenses are generally circularly symmetric
  - Optical center is known

$$\bar{x}' = \bar{x}\left(1 + k_1 r^2 + k_2 r^4 + k_3 r^6 + \ldots\right)$$



Correspondences

- Take picture of known grid – crossings



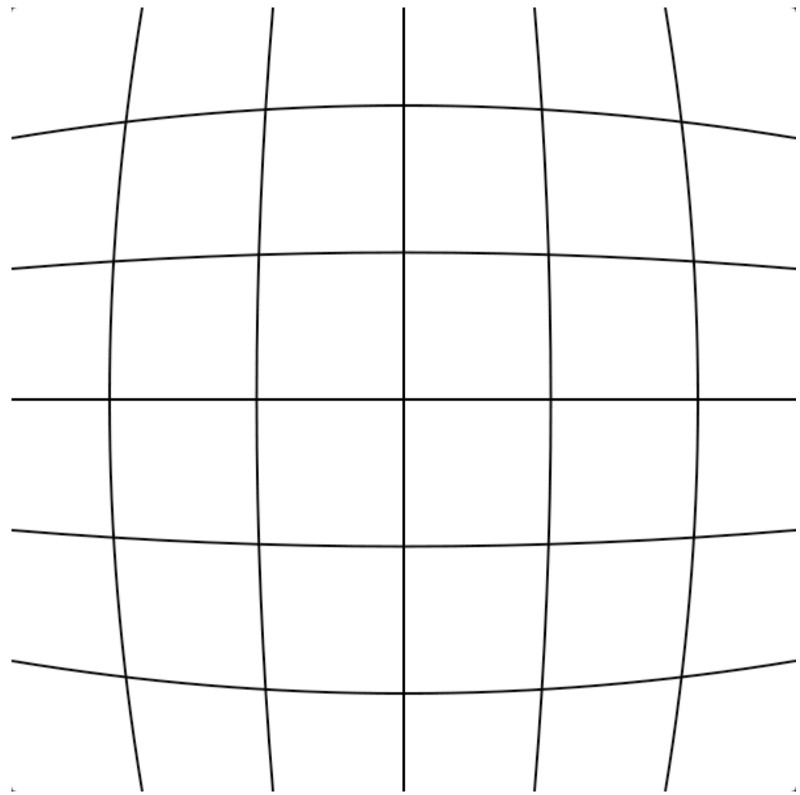


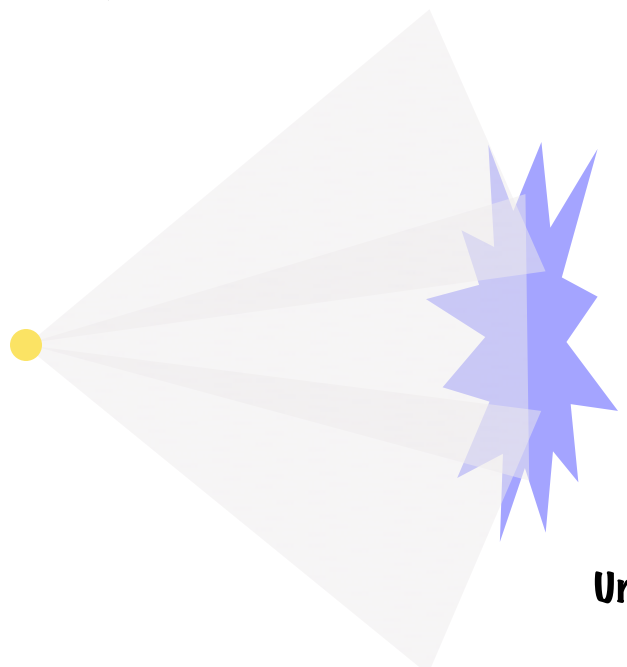
Image Mosaicing

- Piecing together images to create a larger mosaic
- Doing it the old fashioned way
  - Paper pictures and tape
  - Things don't line up
  - Translation is not enough
- Need some kind of warp
- Constraints
  - Warping/matching two regions of two different images only works when…

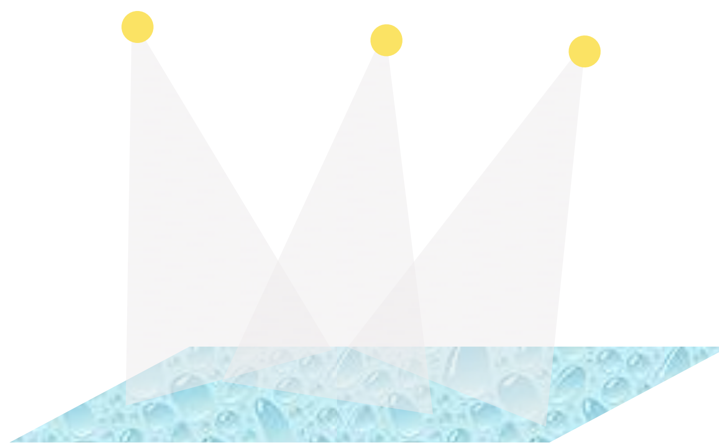
Special Cases

- Nothing new in the scene is uncovered in one view vs another
  - No ray from the camera gets behind another

1) Pure rotations-arbitrary scene

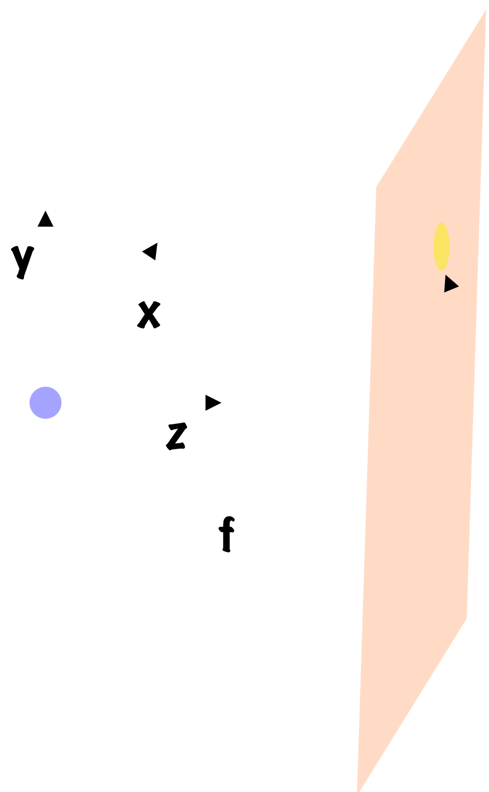


2) Arbitrary views of planar surfaces



3D Perspective and Projection

- Camera model



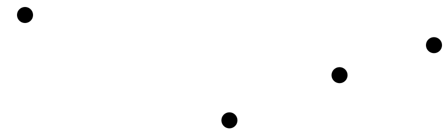
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = f \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix}$$ Image coordinates

Perspective Projection Properties

- Lines to lines (linear)



- Conic sections to conic sections

- Convex shapes to convex shapes

- Foreshortening

Image Homologies

- Images taken under cases 1,2 are perspectively equivalent to within a linear transformation
  - Projective relationships - equivalence is

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \equiv \begin{pmatrix} d \\ e \\ f \end{pmatrix} \Longleftrightarrow \begin{pmatrix} a/c \\ b/c \\ 1 \end{pmatrix} = \begin{pmatrix} d/f \\ e/f \\ 1 \end{pmatrix}$$

Transforming Images To Make Mosaics

Linear transformation with matrix P

$$\bar{x}^* = P\bar{x} \qquad P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & 1 \end{pmatrix} \qquad \begin{aligned} x^* &= p_{11}x + p_{12}y + p_{13} \\ y^* &= p_{21}x + p_{22}y + p_{23} \\ z^* &= p_{31}x + p_{32}y + 1 \end{aligned}$$

Perspective equivalence

$$\begin{aligned} x' &= \tfrac{p_{11}x + p_{12}y + p_{13}}{p_{31}x + p_{32}y + 1} \\ y' &= \tfrac{p_{21}x + p_{22}y + p_{23}}{p_{31}x + p_{32}y + 1} \end{aligned}$$

Multiply by denominator and reorganize terms

$$\begin{aligned} p_{31}xx' + p_{32}yx' - p_{11}x - p_{12}y - p_{13} &= -x' \\ p_{31}xy' + p_{32}yy' - p_{21}x - p_{22}y - p_{23} &= -y' \end{aligned}$$

Linear system, solve for P

$$\begin{pmatrix} -x_1 & -y_1 & -1 & 0 & 0 & 0 & x_1x_1' & y_1x_1' \\ -x_2 & -y_2 & -1 & 0 & 0 & 0 & x_2x_2' & y_2x_2' \\ & & & \vdots & & & & \\ -x_N & -y_N & -1 & 0 & 0 & 0 & x_Nx_N' & y_Nx_2' \\ 0 & 0 & 0 & -x_1 & -y_1 & -1 & x_1y_1' & y_1y_1' \\ 0 & 0 & 0 & -x_2 & -y_2 & -1 & x_2y_2' & y_2y_2' \\ & & & \vdots & & & & \\ 0 & 0 & 0 & -x_N & -y_N & -1 & x_Ny_N' & y_Ny_N' \end{pmatrix} \begin{pmatrix} p_{11} \\ p_{12} \\ p_{13} \\ p_{21} \\ p_{23} \\ p_{23} \\ p_{31} \\ p_{32} \end{pmatrix} = \begin{pmatrix} -x_1' \\ -x_2' \\ \vdots \\ -x_N' \\ -y_1' \\ -y_2' \\ \vdots \\ -y_N' \end{pmatrix}$$

Image Mosaicing

4 Correspondences

5 Correspondences

6 Correspondences

Mosaicing Issues

- Need a canvas (adjust coordinates/origin)
- Blending at edges of images (avoid sharp transitions)
- Adjusting brightnesses
- Cascading transformations

Specifying Warps - Another Strategy

- Let the # DOFs in the warp equal the # of control points (x1/2)
  - Interpolate with some grid-based interpolation
    - E.g. binlinear, splines

Landmarks Not On Grid

- Landmark positions driven by application
- Interpolate transformation at unorganized correspondences
  - *Scattered data interpolation*
- How do we do scattered data interpolation?
  - Idea: use kernels!
- *Radial basis functions*
  - Radially symmetric functions of distance to landmark

RBFs - Formulation

- Represent f as weighted sum of basis functions

$$f(\bar{x}) = \sum_{i=1}^{N} k_i \phi_i(\bar{x}) \qquad \phi_i(\bar{x}) = \phi\left(||\bar{x} - \bar{x}_i||\right)$$

Sum of radial basis functions

Basis functions centered
at positions of data

- Need interpolation for vector-valued function, T:

$$\begin{aligned} T^x(\bar{x}) &= \textstyle\sum_{i=1}^{N} k_i^x \phi_i(\bar{x}) \\ T^y(\bar{x}) &= \textstyle\sum_{i=1}^{N} k_i^y \phi_i(\bar{x}) \end{aligned}$$

Solve For k's With Landmarks as Constraints

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} k_1^x \\ k_2^x \\ \vdots \\ k_N^x \\ k_1^y \\ k_2^y \\ \vdots \\ k_N^y \end{pmatrix} = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_N \\ y'_1 \\ y'_2 \\ \vdots \\ y'_N \end{pmatrix}$$

$$B = \begin{pmatrix} \phi_1(\bar{x}_1) & \phi_2(\bar{x}_1) & \ldots & \phi_N(\bar{x}_1) \\ \phi_1(\bar{x}_2) & \phi_2(\bar{x}_2) & \ldots & \phi_N(\bar{x}_2) \\ \vdots & & & \\ \phi_1(\bar{x}_N) & \phi_2(\bar{x}_N) & \ldots & \phi_N(\bar{x}_N) \end{pmatrix}$$

Issue: RBFs Do Not Easily Model Linear Trends

RBFs - Formulation w/Linear Term

- Represent f as weighted sum of basis functions and

$$f(\bar{x}) = \sum_{i=1}^{N} k_i \phi_i(\bar{x}) + p_2 y + p_1 x + p_o \qquad \phi_i(\bar{x}) = \phi\left(||\bar{x} - \bar{x}_i||\right)$$

Linear part of transformation

Sum of radial basis functions

Basis functions centered at positions of data

- Need interpolation for vector-valued function, T:

$$\begin{aligned} T^x(\bar{x}) &= \sum_{i=1}^{N} k_i^x \phi_i(\bar{x}) + p_2^x y + p_1^x x + p_o^x \\ T^y(\bar{x}) &= \sum_{i=1}^{N} k_i^y \phi_i(\bar{x}) + p_2^y y + p_1^y x + p_o^y \end{aligned}$$

RBFs - Solution Strategy

- Find the k's and p's so that f() fits at data points
  - The k's can have no linear trend (force it into the p's)
- Constraints -> linear system

$$T^x(\bar{x}_i) = x'_i \qquad T^y(\bar{x}_i) = y'_i$$

Correspondences must match

$$\sum_{i=1}^{N} k_i^x = 0 \qquad \sum_{i=1}^{N} k_i^y = 0$$

$$\sum_{i=1}^{N} k_i^x \bar{x}_i = \bar{0} \qquad \sum_{i=1}^{N} k_i^y \bar{x}_i = \bar{0}$$

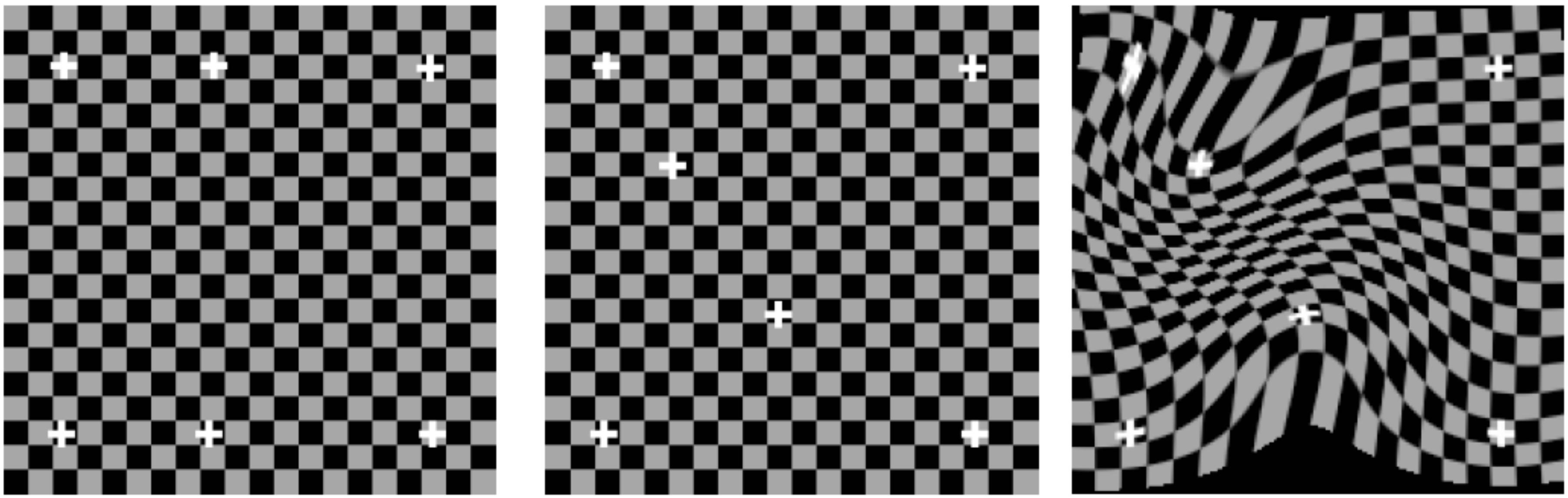
Keep linear part separate from deformation

RBFs - Linear System

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} k_1^x \\ k_2^x \\ \vdots \\ k_N^x \\ p_2^x \\ p_1^x \\ p_0^x \\ k_1^y \\ k_2^y \\ \vdots \\ k_N^y \\ p_2^y \\ p_1^y \\ p_0^y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ x'_1 \\ x'_2 \\ \vdots \\ x'_N \\ 0 \\ 0 \\ 0 \\ y'_1 \\ y'_2 \\ \vdots \\ y'_N \end{pmatrix}$$

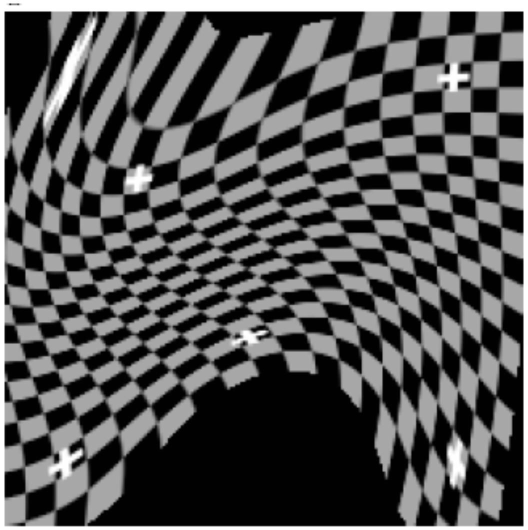
$$B = \begin{pmatrix} x_1 & x_2 & \dots & x_N & 0 & 0 & 0 \\ y_1 & y_2 & \dots & y_N & 0 & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 & 0 \\ \phi_{11} & \phi_{12} & \dots & \phi_{1N} & y_1 & x_1 & 1 \\ \phi_{21} & \phi_{22} & \dots & \phi_{2N} & y_2 & x_2 & 1 \\ \vdots & & & & & & \\ \phi_{N1} & \phi_{N2} & \dots & \phi_{NN} & y_N & x_N & 1 \end{pmatrix}$$

RBF Warp - Example

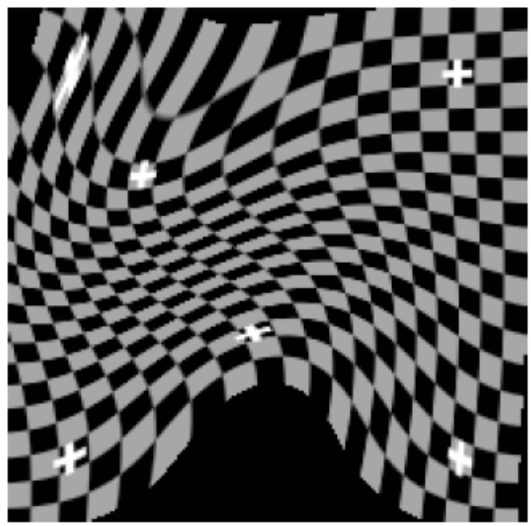


What Kernel Should We Use

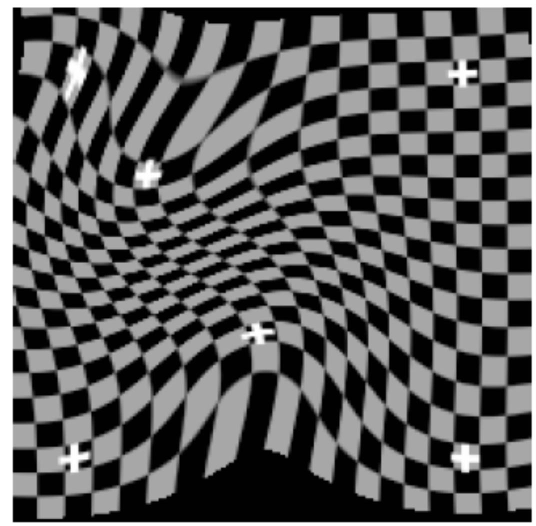
- Gaussian
  - Variance is free parameter - controls smoothness of warp



$\sigma = 2.5$



$\sigma = 2.0$



$\sigma = 1.5$

From: Arad et al. 1994

RBFs - Aligning Faces

Mona Lisa - Target

Venus - Source

Venus - Warped

RBFs - Special Case: Thin Plate Splines

- A special class of kernels

$$\phi_i(x) = ||x - x_i||^2 \lg\left(||x - x_i||\right)$$

- Minimizes the distortion function (bending energy)

$$\int \left[\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)^2 \right] dx dy.$$

  - No *scale* parameter. Gives *smoothest* results
  - Bookstein, 1989

Application: Image Morphing

- Combine shape and intensity with time parameter t
  - Just blending with amounts t produces "fade"

$$I(t) = (1-t)I_1 + tI_2$$

  - Use control points with interpolation in t

$$\bar{c}(t) = (1-t)\bar{c}_1 + t\bar{c}_2$$

  - Use c_1, c(t) landmarks to define T_1, and c_2,c(t) landmarks to define T_2

Image Morphing

- Create from blend of two warped images

$$I_t(\bar{x}) = (1-t)I_1\left(T_1(\bar{x})\right) + tI_2\left(T_2(\bar{x})\right)$$

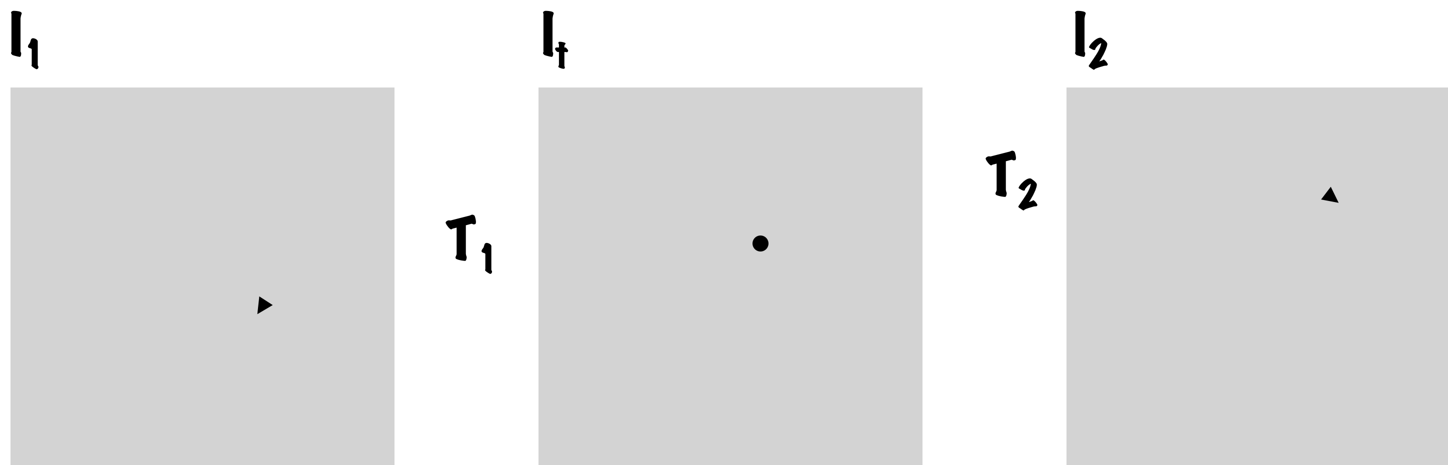


Image Morphing

Application: Image Templates/ Atlases

- Build image templates that capture statistics of class of images
  - Accounts for shape and intensity
  - Mean and variability
- Purpose
  - Establish common coordinate system (for comparisons)
  - Understand how a particular case compares to the general population

Templates - Formulation

- N landmarks over M different subjects/samples

Correspondences

Images

$$I^j(\bar{x}) \qquad \bar{c}_i^j \qquad \begin{pmatrix} \bar{c}_1^1 & \dots & \bar{c}_N^1 \\ \vdots & & \vdots \\ \bar{c}_1^M & \dots & \bar{c}_N^M \end{pmatrix}$$

Mean of correspondences (template)

$$\hat{c}_i = \frac{1}{M} \sum_{j=1}^{M} \bar{c}_i^j$$

Transformations from mean to subjects

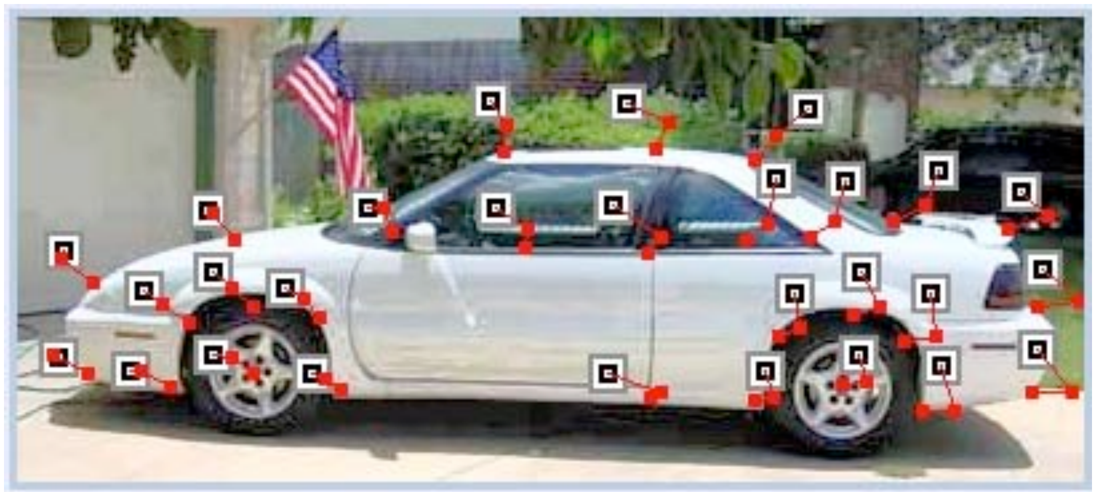
$$\bar{c}_i^j = T^j(\hat{c}_i)$$

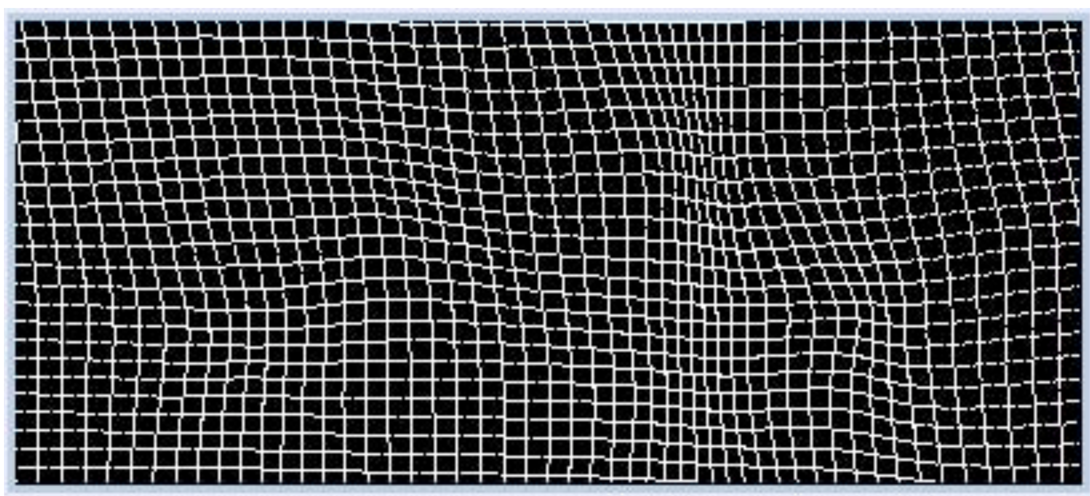
Templated image

$$\hat{I}(\bar{x}) = \frac{1}{M} \sum_j I^j\left(T^j(\bar{x})\right)$$

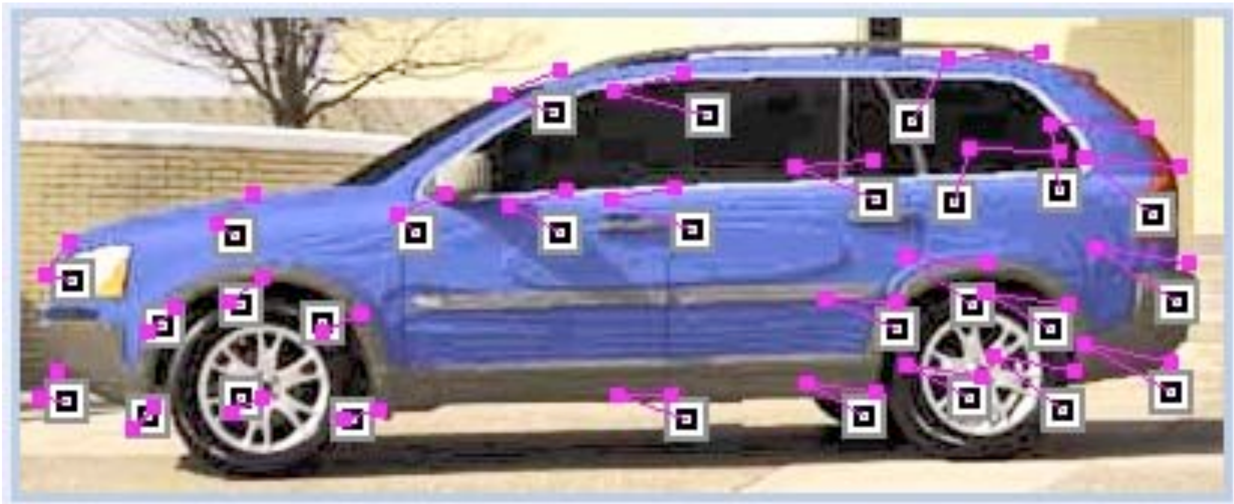
Cars

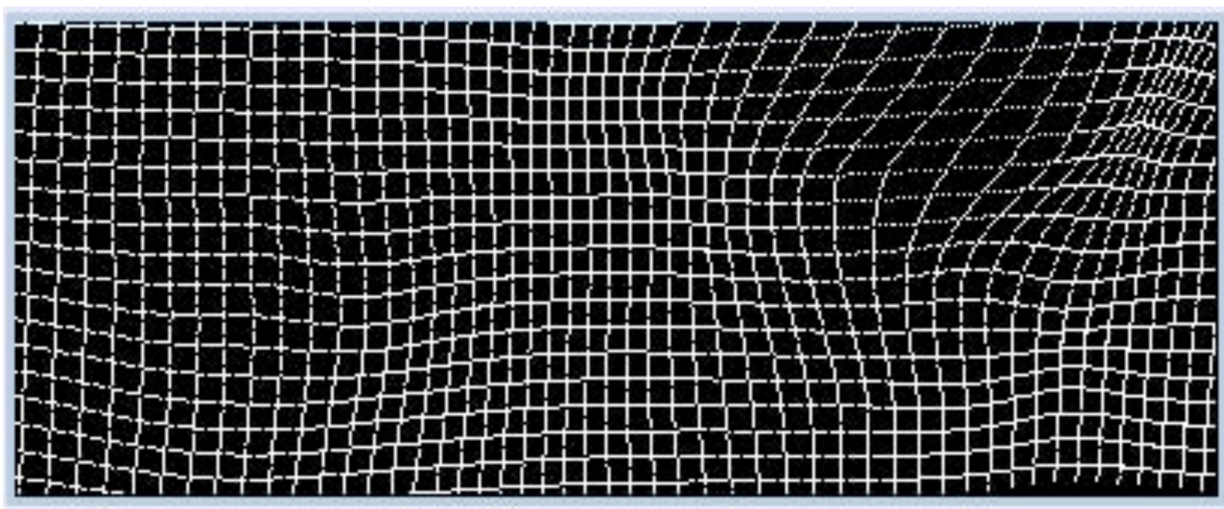
Car Landmarks and Warp



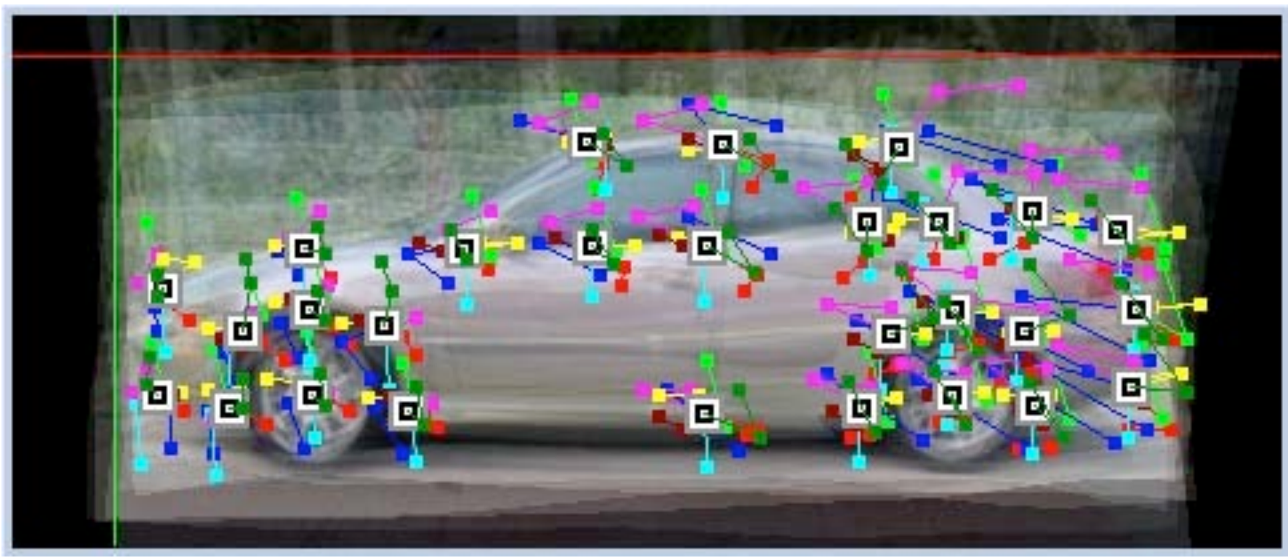


Car Landmarks and Warp

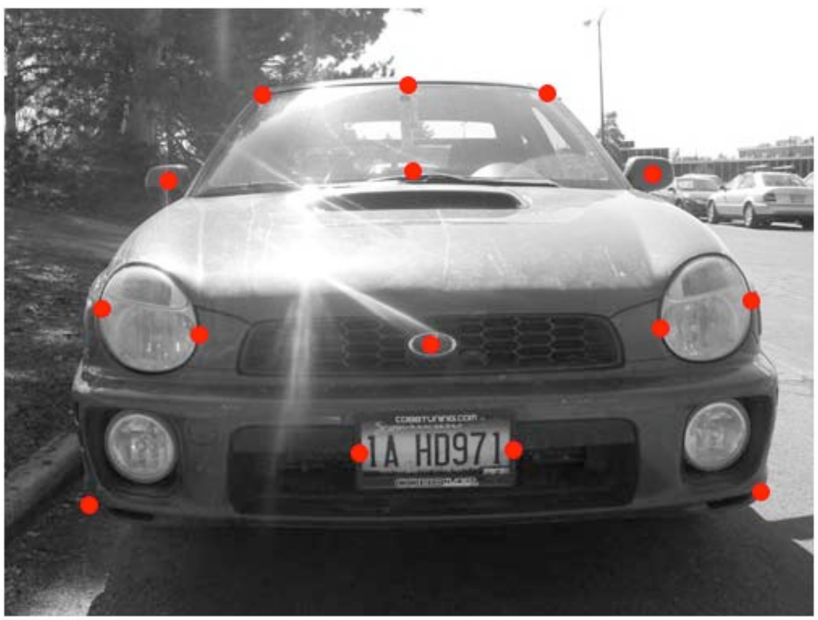




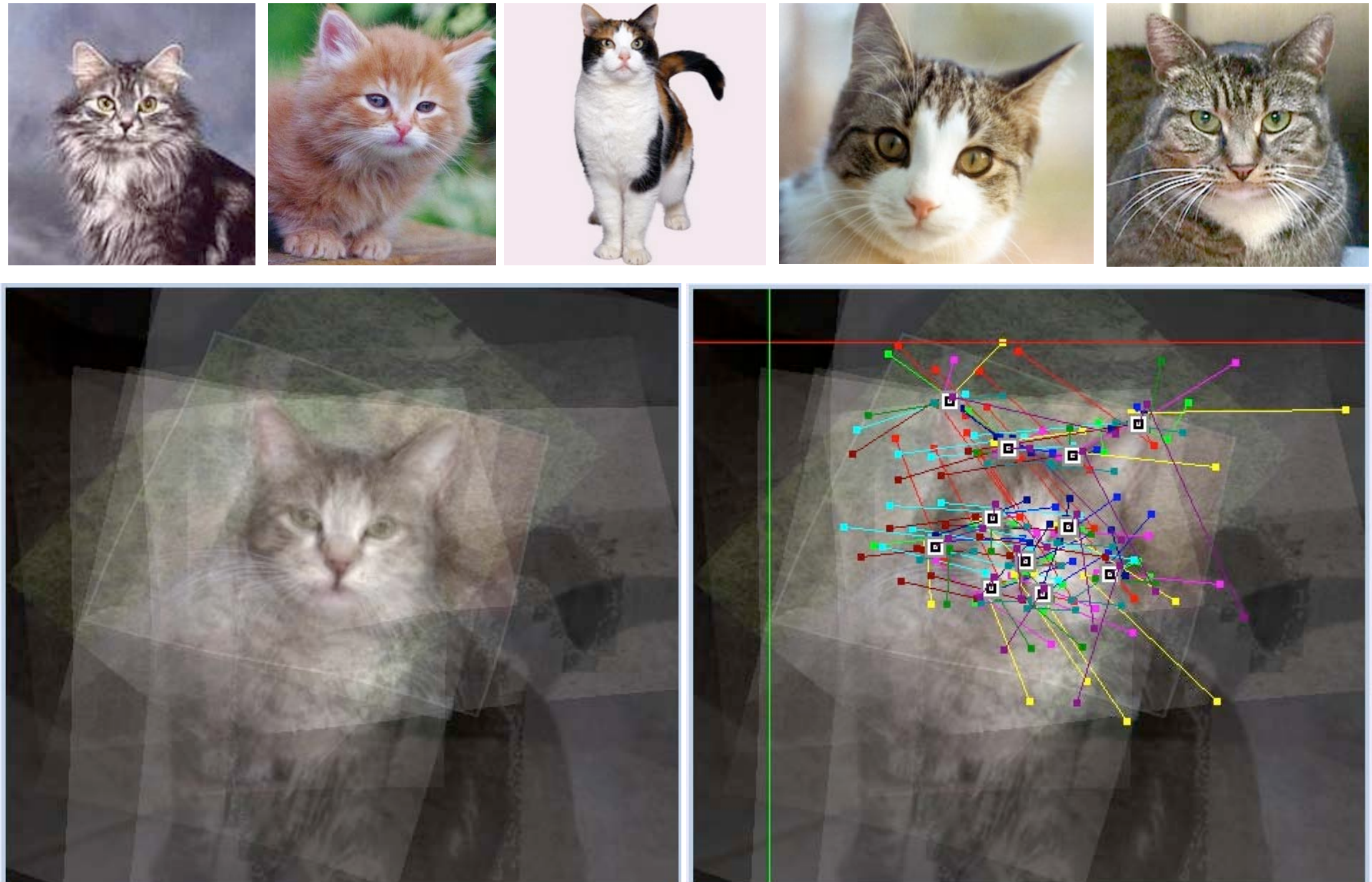
Car Mean



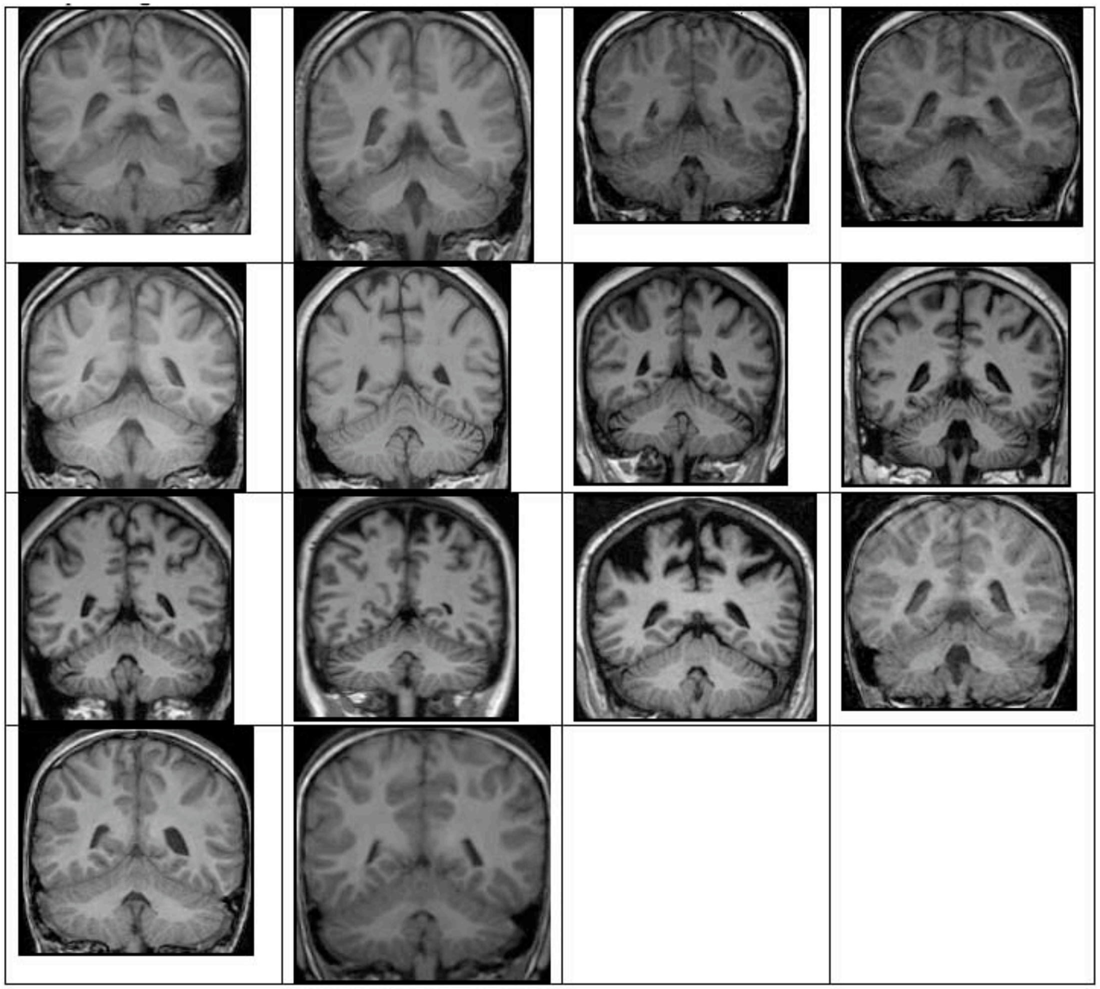
Cars



Cats



Brains



Brain Template