CS 6220 Spring 2011
Homework Set Five
Differential Equations

I. Discretization of Ordinary Differential Equations

There is an almost legendary system of ordinary differential equations:

$$\begin{aligned} \frac{dx}{dt} &= Bx - Cxy \\ \frac{dy}{dt} &= -Dy + C\kappa xy \end{aligned} \tag{1}$$

known as the Lotka-Volterra system, used to model population dynamics. Concentration of prey x multiplies in isolation, while being consumed by predator having concentration y. Positive constants are birth rate B for x, measure of the frequency that predator catches prey constant C, death rate for y due to lack of prey constant D, and κ (warning: this is gruesome!), a share constant matching how many unit predators it takes to devour a single prey. Vito Volterra had the idea in 1926 to set Darwin's idea of "survival of the fittest" to differential equations in order to describe fish and shark populations, while Alfred Lotka used these equations in 1925 to describe oscillations in chemical reactions. (For your information, *lotke* literally means "potato pancake" in Yiddish.) Lotka-Volterra attempts to accurately quantify Darwin's principle of natural selection as a physical law. Lotka's idea was that evolution favored selecting according to the " maximum useful energy flow transformation."

Consider a system with $\kappa = 1$, $B = .4$, $C = .5$ and $D = .45$, having initial conditions $x(t = 0) = .1$ and $y(t = 0) = .2$. Plot $x(t)$ and $y(t)$ for t long enough you can judge the asymptotic behavior as $t \to \infty$. Your plots should show x and y separately and together in a "phase portrait" in the plane of the trajectory $(x(t), y(t))$.

1. Provide a physical interpretation of the observed behavior.

2. Experiment with other initial populations where one or the other x and y populations eventually becomes extinct.

3. Find nonzero initial populations that never change.

4. You may, if you wish, experiment varying parameter constants κ, B, C, D.

You should integrate (1) using *third order Adams-Bashford* iteratively given by

$$U_{i+1} = U_i + h((23/12)F_i - (4/3)F_{i-1} + (5/12)F_{i-2}) \tag{2}$$

where h is $t_{i+1} - t_i$, U is the column vector of x and y, and F is the right hand side of (1). Subscripts refer to evaluation at time t_i.

Note that with U_0 given, we are unable to evaluate U_1 and U_2 from (2). We recommend you use Heun's (second order) method which will ensure that the values obtained for x_1, y_1, x_2 and y_2 fall within the Adams-Bashford region of stability. Heun's method has the form

$$U_{i+1} = U_i + \frac{h}{2}(F_i + \tilde{F}_i)$$

where $\tilde{F}_i = F(t_{i+1}, U_i + hF_i)$.

II. Forward Difference Discretization of the Heat Equation

The *parabolic* partial differential equation

$$u_t = u_{xx}, \quad 0 < x < 1, \quad 0 \leq t, \tag{3}$$

subject to the initial condition

$$u(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1,$$

and Dirichlet boundary conditions

$$u(0, t) = u(1, t), \quad 0 < t,$$

is an example of the heat or diffusion equation. It has exact solution

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x).$$

The *Forward Difference Method* applied to (3) yields the recurrence

$$u_i^{k+1} = u_i^k + \frac{\Delta t}{(\Delta x)^2} \left(u_{i+1}^k - 2u_i^k + u_{i-1}^k \right), \; i = 1, \ldots, n\,,$$

where spatial grid point x_i equals $i\Delta x$, $i = 0, 1 \ldots, n+1$, with $\Delta x = 1/(n+1)$, temporal grid point t_k equals $k\Delta t$, $k = 0, 1, \ldots$, and where u_i^k denotes the approximate solution to $u(x_i, t_k)$ at grid point (x_i, t_k).

1. Using FDM solve (3) with $\Delta x = .1$ and $\Delta t = .0005$.

2. Plot the computed solution as a three-dimensional surface over the (x, t) plane.

3. Determine the maximum error in your computed solution.

4. Experiment with various spatial grid sizes Δx and characterize the error as a function of Δx by plotting on a log-log scale the maximum error as a function of Δx.

In order that your solution satisfy

Consistency: the local truncation error goes to zero
Stability: approximate solutions remain bounded

a very pretty (and not difficult) analysis shows that

$$\Delta t \leq \frac{(\Delta x)^2}{2}$$

must be heeded as you experiment with different Δx.

5. Repeat 1., with $\Delta x = .1$ and $\Delta t = .01$. Please discuss the observed behavior and possible reasons for its occurrence.