Image Denoising Algorithms

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Abstract. This is a report of an assignment of the class Mathematics of Imaging. In this assignment, we first implement different image denoising algorithms. Namely, H1 regularization, Total Variation (TV) Primal form, TV Dual form, and TV Primal-Dual form, and then we test and compare these algorithms on different images.

Keywords: Image Denoising, H1 Regularization, Total Variation

1 Introduction

Image restoration is an image processing step, in which both the input and output are images. In image restoration, one tries to improve the quality of an image. For example, remove the noise from an image, make a blurred image sharper, or fill some missing portion of an image. It's a important process since it usually improves the performance of other image processing step, such as image segmentation and image registration.

Image denoising is a research field belonging to image restoration, which tries to remove the noise from the image. There are different approaches to accomplish this job, one of them is the variational approach, which tries to minimize a functional.

In this report, we will discuss different algorithms of the variational approach. The first one is called H1 regularization algorithm. This algorithm want to minimize the L_2 norm of the gradient of the image. It has some good properties, for example, it has unique solution, and it is easy to implement, but it does not accept contour discontinuities, causing the obtained solution to be smooth. Rudin, Osher, and Fatemi (ROF) introduced total variation based methods in order to preserve sharp edges. While the ROF model can preserve edges, it also has some numerical drawbacks. A lot of researchers proposed many different solutions, and we will discuss some of them, such as, total variation dual form, and total variation primal-dual form.

2 Methods

The image denoising problem can be formulated as the following. Given an observed image f, we know f is the addition of the ideal image u and some noise with mean 0 and variance σ^2. Our goal is to compute the ideal image u, which should satisfy the constraint,

$$\|u - f\|_2^2 \leq |\Omega|\sigma^2,$$

The problem is underdetermined and obviously an ill posted problem, since given f and σ, there exists infinite u satisfying the above constraint.

The usual method to solve a underdetermined system is to add more constraints to the system. Usually, the constraint can be represented as minimizing a function $g(u)$. For example, if we prefer the $u^\star$ with minimum $l_p norm$ among all possible u. It can be represented as $\min_u \quad g(u) = \min_u \quad \int_\Omega \|u\|_p$. There might be several $u^\star$ with the minimum norm, depending what the constraint is, and which norm is used. In addition, different constraints mean different preference, and therefore adding different constraints will probably obtain different $u^\star$.

Now, the problem becomes

$$\begin{aligned} \min_u \quad & g(u) \\ \text{s.t.} \quad & \|u - f\|_2^2 \leq |\Omega|\sigma^2, \end{aligned} \tag{1}$$

This is an constrained problem. Using Lagrange multiplier, it's equivalent to the following unconstrained minimization problem with appropriate λ, which is a variable.

$$\min_u \quad g(u) + \lambda(\|u - f\|_2^2 - |\Omega|\sigma^2) \tag{2}$$

λ is determined by σ. In other words, there is a one to one corresponding between λ and σ. If we know the standard deviation σ, we will know λ. We will be able to solve both the constraint problem (1) and the unconstrained problem (2). But the problem is that σ is usually unknown, so we can not solve neither the constraint problem (1) nor the unconstraint problem (2).

However, observing that the above minimization problem is equivalent to the following problem, if λ is known.

$$\min_u \quad g(u) + \lambda\|u - f\|_2^2 \tag{3}$$

We can solve this problem by the following two steps. First, we pick a λ, and then we will solve the minimization problem (3). The value we pick for λ is important, so usually we need to test different λ and find the best one.

Now, we will consider the constraint function $g(u)$. As mentioned before, different constraints have different meaning. One type of the suitable choice of the constraint is about the gradient of the image u. For example,in H1 regularization, $g(u) = \int_\Omega |\nabla_x u|^2 dx$; in Total Variation Primal form, $g(u) = \int_\Omega |\nabla_x u| dx$. We will list the equations we want to solve for each of different constraints and discuss the performance of each algorithm in the next section.

In H1 regularization, the functional we want to minimize is

$$\min_u \quad \int_\Omega |\nabla_x u|^2 dx + \lambda \|u - f\|_2^2,$$

We arrive at the Euler-Lagrange equations

$$\begin{aligned} & -\Delta u + \lambda(u - f) = 0 \quad \text{on} \Omega \\ & \langle \nabla u, v \rangle = 0 \quad on \quad \partial\Omega \end{aligned}$$

In Total Variation Primal form, the functional we want to minimize is

$$\min_u \quad \int_\Omega |\nabla_x u| dx + \lambda \|u - f\|_2^2,$$

We arrive at the Euler-Lagrange equations

$$\begin{aligned} & -\operatorname{div}(\frac{\nabla u}{|\nabla u|}) + 2\lambda(u - f) = 0 \quad \text{on} \Omega \\ & \langle \nabla u, v \rangle = 0 \quad on \quad \partial\Omega \end{aligned}$$

The Dual form of the Total Variation [2]. We have the following equivalent forms

$$\int_\Omega |\nabla_x u| = \max_{\omega \in C_0^1(\Omega), |\omega| \le 1} \quad \int_\Omega \nabla_x u \cdot \omega = \max_{|\omega| \le 1} \quad \int_\Omega -u \operatorname{div} \omega,$$

Therefore, the functional we want to minimize is

$$\min_u \quad \max_{\omega \in C_0^1(\Omega), |\omega| \le 1} \quad \int_\Omega -u \operatorname{div} \omega + \lambda \|u - f\|_2^2. \tag{4}$$

Using min-max theorem, we can solve this problem by first minimizing

$$\min_{\omega \in C_0^1(\Omega), |\omega| \le 1} \quad \frac{1}{2} \| \operatorname{div} \omega + 2\lambda f \|_2^2,$$

and then, compute u as $u = f + \frac{1}{2\lambda} \operatorname{div} \omega$.

The Total Variation Primal-Dual [1] is just a different numerical algorithm to solve the dual form (4).

3 Results and Discussion

In this section, we will compare and discuss the results of the different algorithms.

3.1 H1 regularization

Fig. 1. First row: Original and noisy image; Second row: two H1 restorations with different λ.

In H1 regularization, $g(u) = \int_{\Omega} |\nabla_x u|^2 dx$, and we want to minimize the L^2 norm of the gradient of the image, which is very large in an image with edges. So the H1 regularization algorithm will blur the edges as shown in Figure 1.

From Figure 1, we can also see that different λ will result in different restoration. This is easy to understand, since different λ allow different level of variance of the noise.

3.2 TV Primal form

There is an unstable operation in the TV Primal from, which is $\mathrm{div}(\frac{\nabla u}{|\nabla u|})$. When $|\nabla u|$ is zero, it's unstable. One solution is to add a small constant β to the denominator. In Figure 2, we can see that without β there are some black block in the restored image caused by the unstable operation. By introducing β, the operation is stable, but it also introduces some blurring to the image.

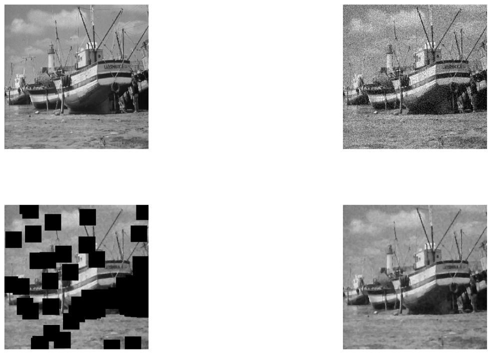

Fig. 2. First row: Original and noisy image; Second row: two TV Primal restorations with (right) and without (left) β.

3.3 TV Dual Form and TV Primal-Dual Form

Since TV Dual Form and TV Primal-Dual Form are different algorithms to solve the same equation, we will compare them together here. From Figure 3, it's hard to tell which algorithm is better. They are roughly the same. The Primal-Dual restorations may be a little bit better than the Dual restorations.

Fig. 3. First column: Original and noisy image; Second column: two TV Dual restorations with different τ. Third column: two TV Primal-Dual restorations with different τ

3.4 Test all algorithms on Fluoroscopy images

In this section, we test all algorithms on two Fluoroscopy images, one has medium noise, the other has severe noise.

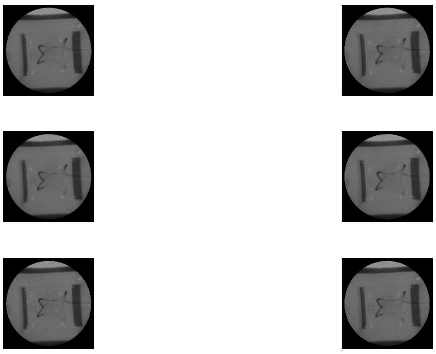

Fig. 4. Fluoroscopy image with medium noise. First row: Original and noisy image; Second row: H1 restoration (left) and TV primal restoration (right). Third row: TV Dual restoration (left) and TV Primal-Dual restoration (right)

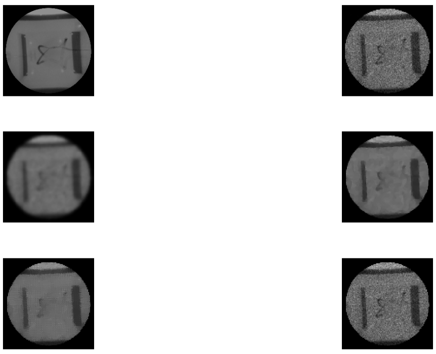

Fig. 5. Fluoroscopy image with severe noise. First row: Original and noisy image; Second row: H1 restoration (left) and TV primal restoration (right). Third row: TV Dual restoration (left) and TV Primal-Dual restoration (right)

3.5 SNR and Computational complexity

In Figure 6, we test the residual between the noise free image and the restored image in each iteration. For TV primal and TV primal-dual algorithms, the residual is decreasing, and TV primal is decreasing faster than TV primal-dual for this image. For H1 regularization, however, the residual is first decreasing but after some time, it starts to increase. I think this is caused by the blurring. We know that H1 regularization will blur the image. At the beginning, H1 regularization blurs the image, but it still improve the quality of the image somehow, but after some iteration, it blurs the image too much and the image quality will decrease.

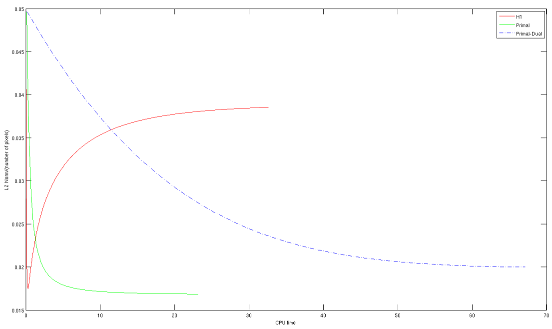

Fig. 6. The X axis is the CPU time (seconds). The Y axis is the L2 norm of the different between the noisy image and noise free image

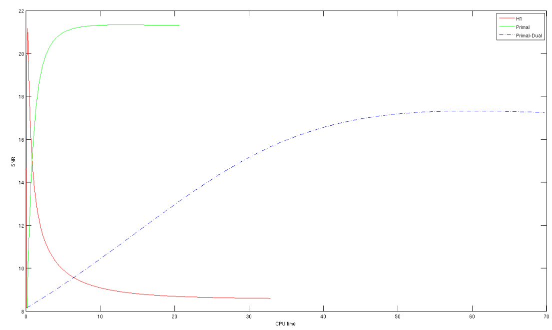

Fig. 7. The X axis is the CPU time (seconds). The Y axis is the SNR

In Figure 6, we compute the SNR of restored image in each iteration. For TV primal and TV primal-dual algorithms, the SNR is increasing, however, for H1 regularization, the SNR is first increasing and then decreasing, which is also caused by the H1 regularization blurring.

References

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