Adaptive Riemannian Metrics for Improved Geodesic Tracking of White Matter

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Geodesic Tracking on DTI

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- A starting point p
- A local cost function $f(x, v)$

$$\min \int_0^1 f(x, v)\,dt$$

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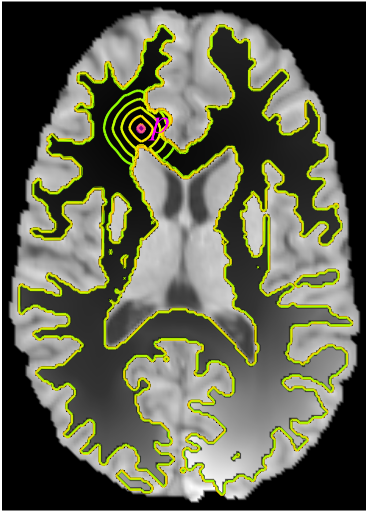

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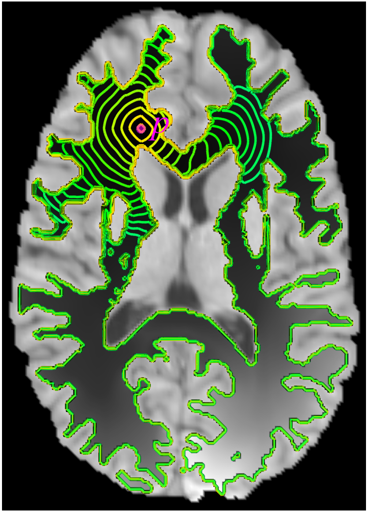

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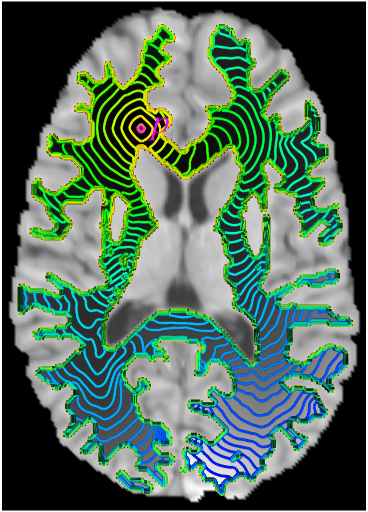

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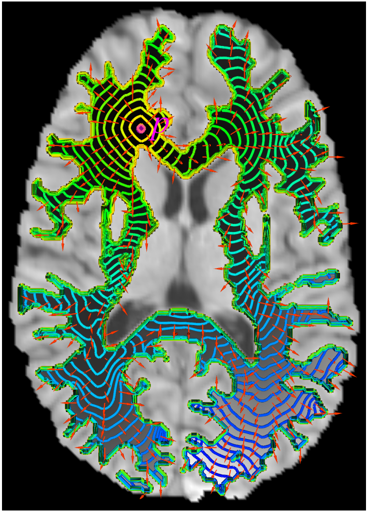

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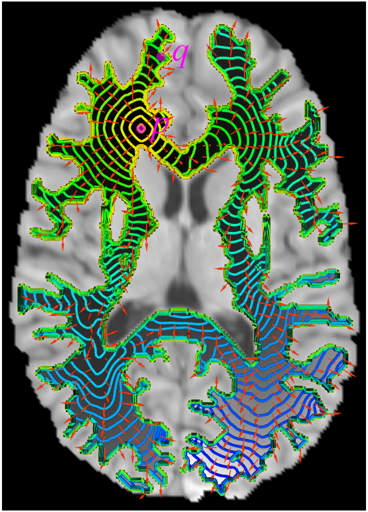

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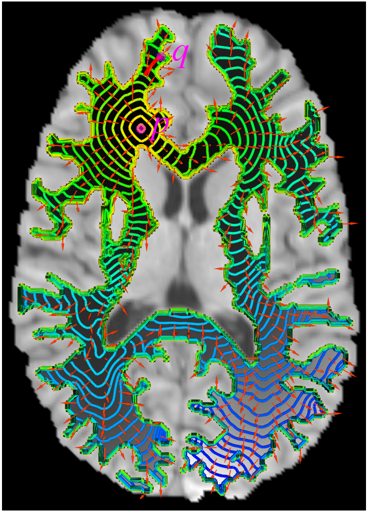

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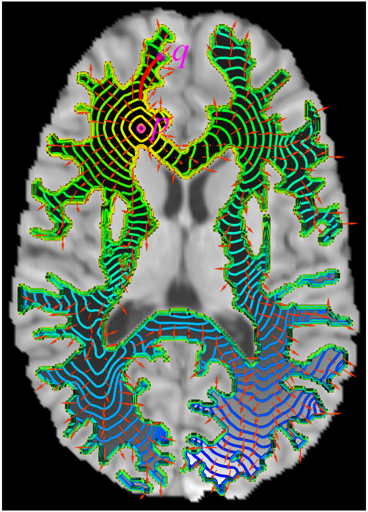

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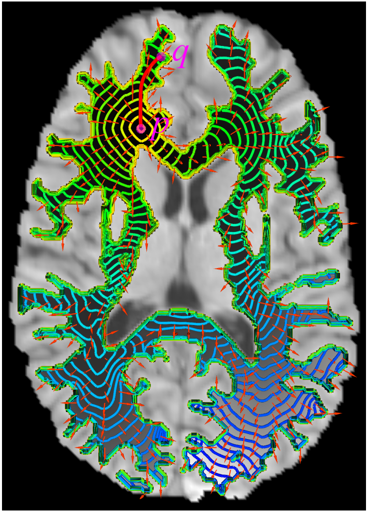

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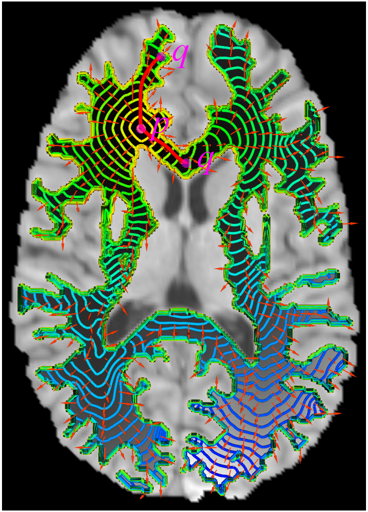

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1. Compute the time-of-arrival function u by propagating from p
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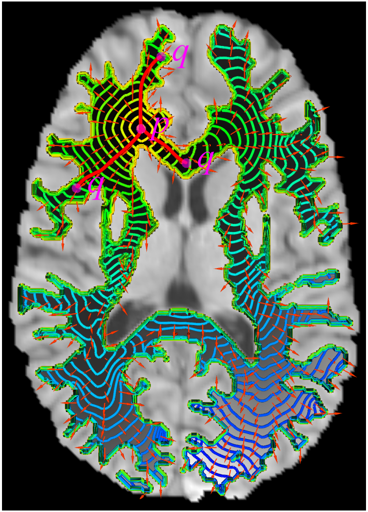
Output:

- Time-of-arrival function u
- The geodesics between p and any other point q



Related Work

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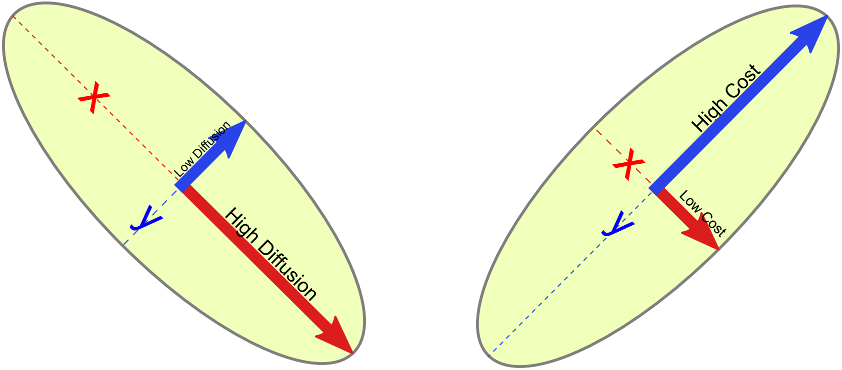
- Jackowski et al. MICCAI 2004
- Pichon et al. MICCAI 2005
- Fletcher et al. IPMI 2007
- Jbabdi et al. IJBI 2008
- Sepasian et al. MICCAI Workshop on Computational Diffusion MRI (CDMRI) 2009

Inverse Diffusion Tensor as Riemannian Metric

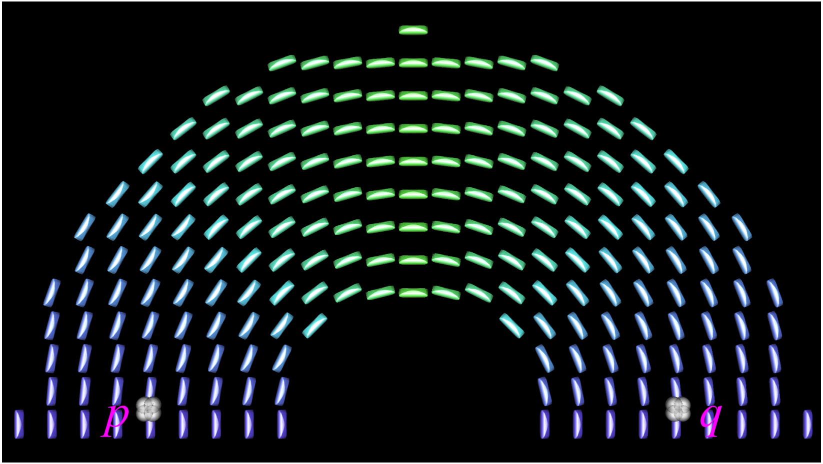
Diffusion Tensor:
$D = R\Sigma R^{-1}$

Riemannian Metric:
$g = D^{-1} = R\Sigma^{-1}R^{-1}$



The corresponding cost function $f = \langle T(t), T(t) \rangle_g = T(t)^T D^{-1} T(t)$.

Motivation: Problem With Inverse Tensor Metric



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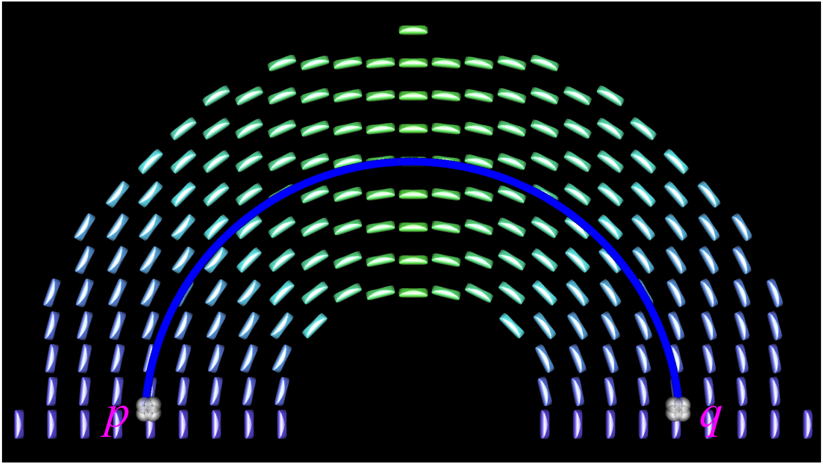

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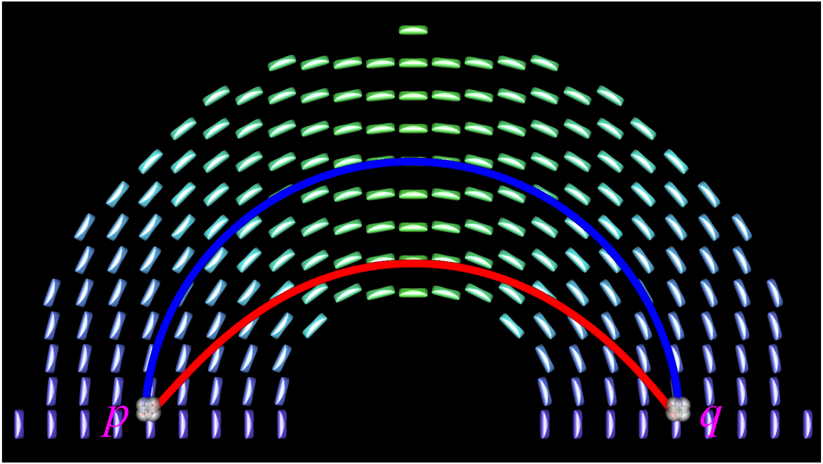

Our Contribution: Improved Geodesic Tracking

Goal

We want the geodesics closely conform to the principal eigenvector field.

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We compute a spatially-varying scalar function,α, that modulates the inverse tensor field at each point.

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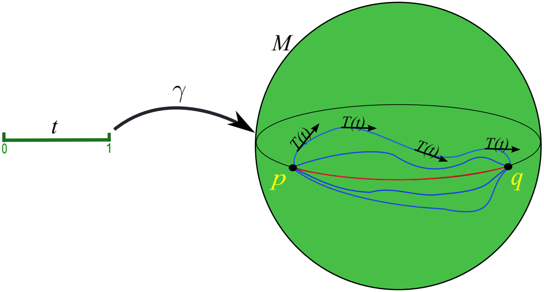
$$g = e^{\alpha} D^{-1}$$

We call the function e^{α} the *metric modulating function*.

Background: Riemannian Manifold and Geodesic



On a Riemannian manifold, M, the geodesic between two points $p, q \in M$ is defined by the minimization of the energy functional

$$E(\gamma) = \int_0^1 \langle T(t), T(t) \rangle_g \, dt.$$

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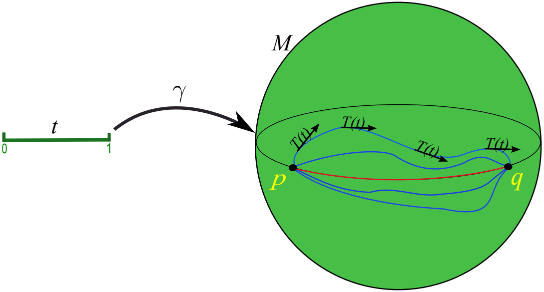

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Euler-Lagrange Equation is

$$\nabla_T T = 0.$$

The symbol ∇ here is affine connection.

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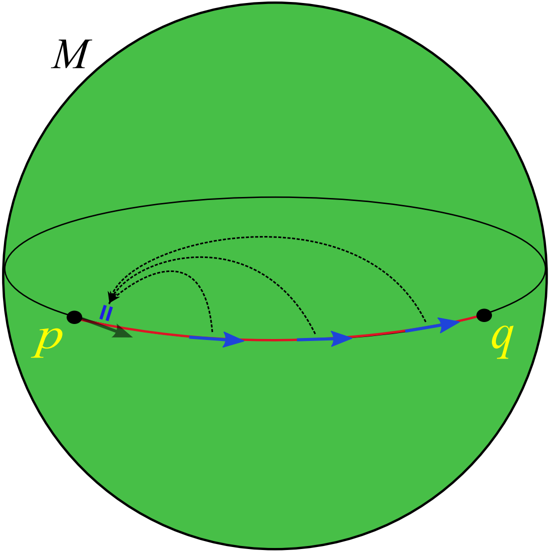
In the special case of $Y = T$,

- $\nabla_T T$ measures how the vector field T bends along its integral curves.
- $\nabla_T T = 0$ means that tangent vectors, T, remain parallel if they are transported along the geodesic.

Background: Affine Connection $\nabla_T T = 0$



Compute the Geodesic by Solving the Eikonal Equation

A geodesic which minimize $E(\gamma(t))$ also satisfies the Eikonal equation.

$$\nabla u^T g^{-1} \nabla u = 1 \quad (1)$$

$$T(t) \propto g^{-1} \nabla u \quad (2)$$

u is the time-of-arrival function from a starting location p.

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u is the time-of-arrival function from a starting location p.

To compute the all geodesics from a starting point p, we need 2 steps:

1. Choose a starting point p, solve the Eikonal equation(1) for u with the initial condition $u(p) = 0$.
2. Compute $T(t)$ from (2) and then integrate $T(t)$ backward to p from any other points.

Geodesic Energy Functional

The traditional geodesic energy functional is

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Now, since we scale the Riemannian metric by a positive function $e^{\alpha(x)}$, which results in the new geodesic energy functional

$$E_\alpha(\gamma) = \int_0^1 e^{\alpha(\gamma(t))} \langle T(t), T(t) \rangle \, dt.$$

Compute the New Geodesic Equation

We take the variational of the energy $E_\alpha(\gamma)$, results in the new geodesic equation

$$\operatorname{grad}\alpha \cdot \|T\|^2 = 2\nabla_T T + 2d\alpha(T) \cdot T,$$

where $\operatorname{grad}\alpha$ is the Riemannian gradient defined as $g^{-1}\left(\frac{\partial\alpha}{\partial x^1}, \frac{\partial\alpha}{\partial x^2}, \ldots, \frac{\partial\alpha}{\partial x^n}\right)$.

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Now taking the inner product with T on both sides of (3), we obtain

$$\begin{aligned} \langle \operatorname{grad}\alpha, T\rangle &= 2d\alpha(T) = 2\langle \operatorname{grad}\alpha, T\rangle \\ \Rightarrow \langle \operatorname{grad}\alpha, T\rangle &= 0 \qquad\qquad \Rightarrow d\alpha(T) = 0 \end{aligned}$$

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Therefore, assuming $\|T\| = 1$, $d\alpha(T)$ must vanish, and we get

$$\operatorname{grad}\alpha = 2\nabla_T T.$$

Compute the Metric Modulating Function e^{α}

The simplified geodesic equation is

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Satisfying this property directly would result in the equation

$$\operatorname{grad} \alpha = 2\nabla_V V,$$

which we would need to solve for α.

Compute the Metric Modulating Function e^{α}

However, given an arbitrary V, there may not exist such a function, α, with the desired gradient field. Instead we minimize the functional

$$F(\alpha) = \int_M \|\operatorname{grad}\alpha - 2\nabla_V V\|^2 \, dx.$$

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Euler-Lagrange Equation is

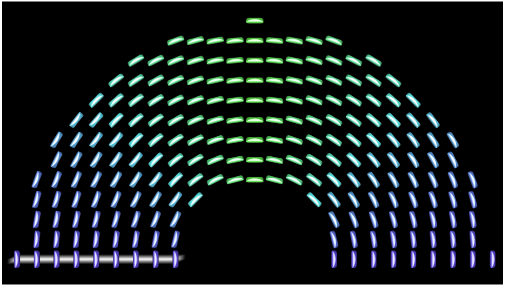
$$\Delta\alpha = 2\,\mathrm{div}\,(\nabla_V V)\,,$$

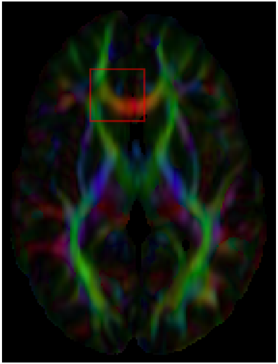
which is a Poisson equation on the Riemannian manifold. $\Delta\alpha$ is the Laplace-Beltrami operator on M.

The appropriate boundary conditions for this problem are the Neumann conditions,

$$\frac{\partial\alpha}{\partial\overrightarrow{n}} = \langle\mathrm{grad}\,\alpha, \overrightarrow{n}\rangle = \langle 2\nabla_V V, \overrightarrow{n}\rangle.$$

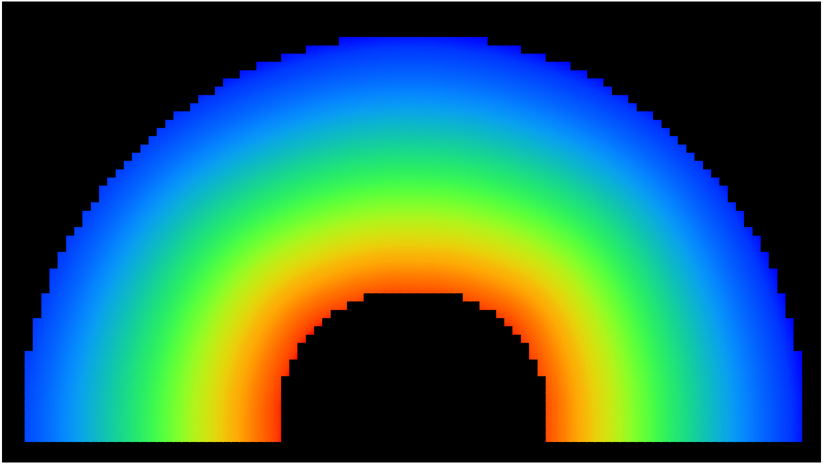
Data





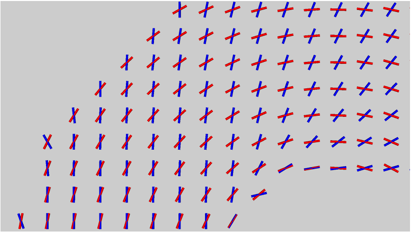
Solution α for the Annulus

$$\alpha(x) = -2\ln r(x) + C \qquad \operatorname{grad}\alpha = 2\nabla_V V$$

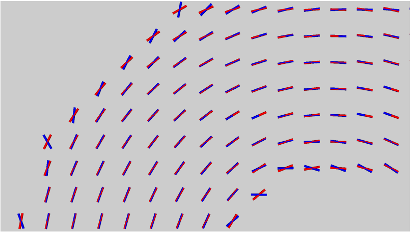


Comparing T with V

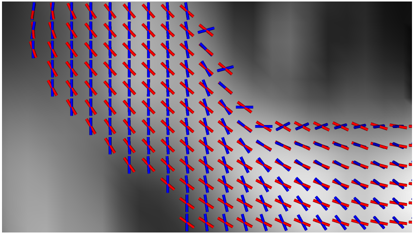
Geodesic Tangent Vectors — Principal Eigenvectors



Inverse Tensor Metric



Our Modulated Metric



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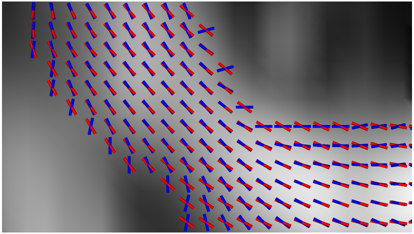
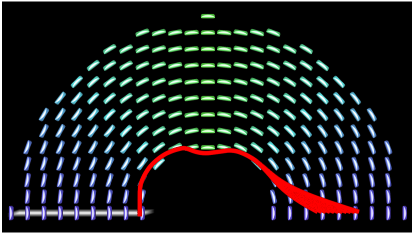


Our Modulated Metric

Geodesics



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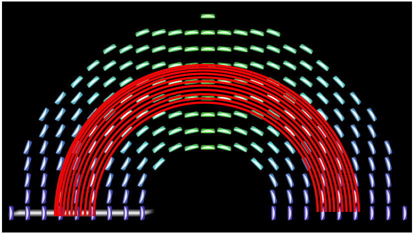

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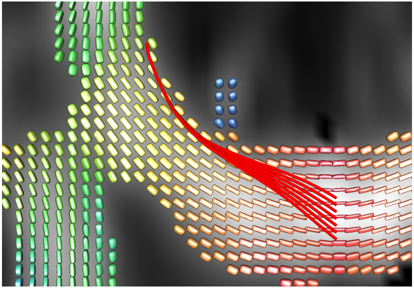

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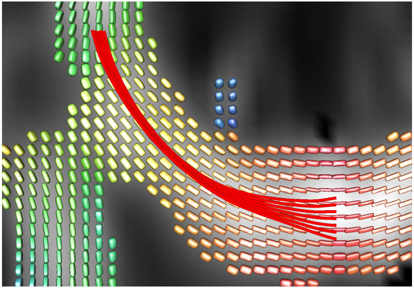
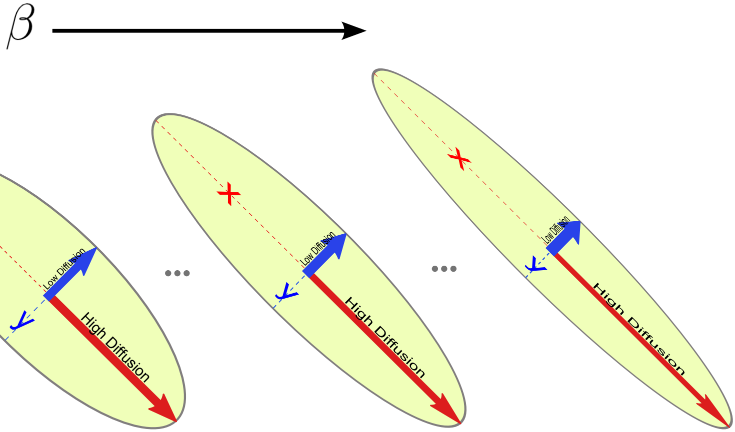


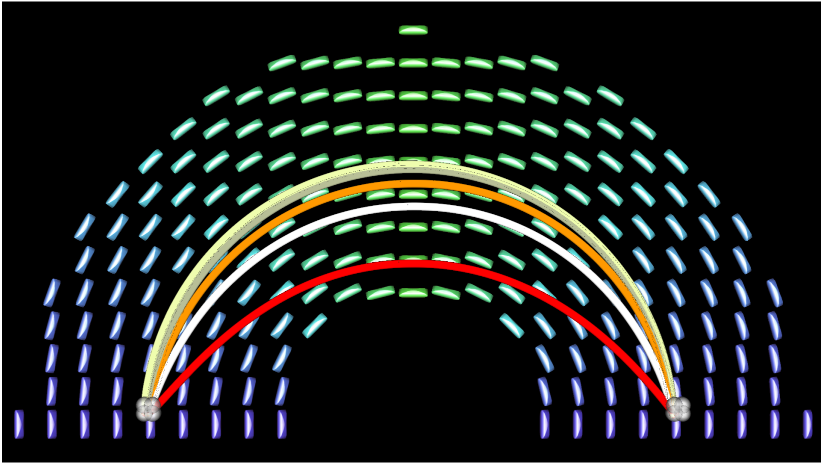
Our Modulated Metric

Sharpened Diffusion Tensor as Riemannian Metric

$$D' = |D|^{\frac{1}{3}} \left(\frac{D}{|D|^{\frac{1}{3}}} \right)^{\beta}, \qquad g = D'^{-1}$$

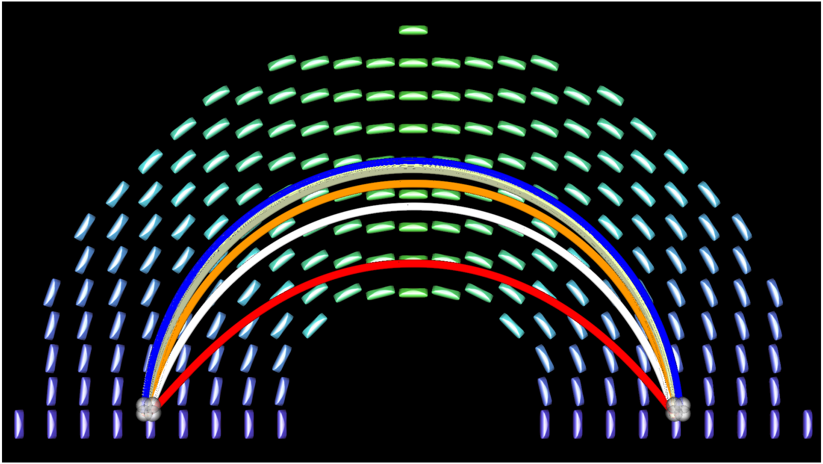


Sharpened Tensor Metric



Geodesics using sharpened tensor metric with different β.

Sharpened Tensor Metric



The blue curve is the ideal geodesic.

Summary and Future Work

To summarize, we compute a spatially-varying scalar function,α, that modulates the inverse tensor metric at each point. The geodesics computed using our modulated metric can faithfully follow the principal eigenvector field.

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Future Work:

- Segment white matter tracts from the computed geodesics.
- Speed up the numerical α solver by parallel computing.
- Solve the edge-crossing issue.
- Extend the proposed method to HARDI.

Thank you!

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