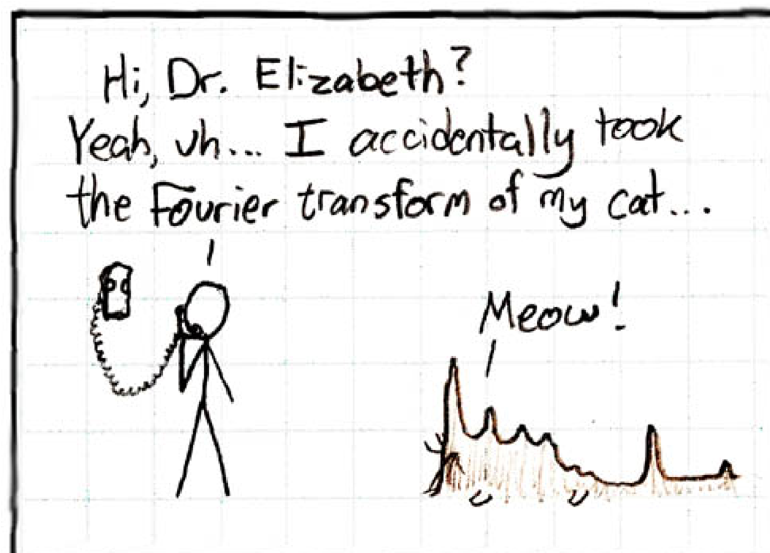
Fourier Transforms

CS 6640 Krithika Iyer

- Fourier Series and Fourier Transforms
- 1D Discrete Fourier Transforms(DFT)
- Fast Fourier Transform(FFT)
- Properties of Fourier Transforms
- 2D DFT for Images



https://xkcd.com/26/

1D Fourier Transform

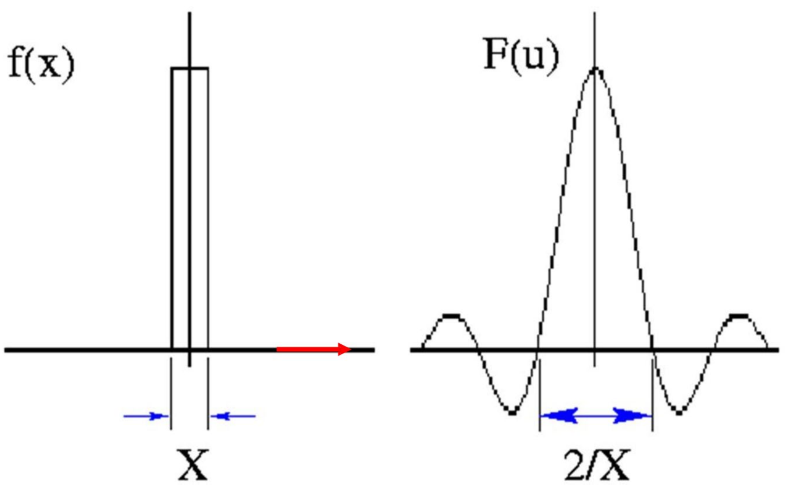
- Reminder transform pair – definition

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux}\, dx,$$
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux}\, du$$

- Example

$$f(x) = \begin{cases} 1, & |x| < \frac{X}{2}, \\ 0, & |x| \geq \frac{X}{2}. \end{cases}$$

$$\begin{aligned} F(u) &= \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx \\ &= \int_{-X/2}^{X/2} e^{-j2\pi ux} dx \\ &= \frac{1}{-j2\pi u}[e^{-j2\pi uX/2} - e^{j2\pi uX/2}] \\ &= X\frac{\sin(\pi X u)}{(\pi X u)} = X\mathrm{sinc}(\pi \mathrm{X} \mathrm{u}). \end{aligned}$$



2D DFT

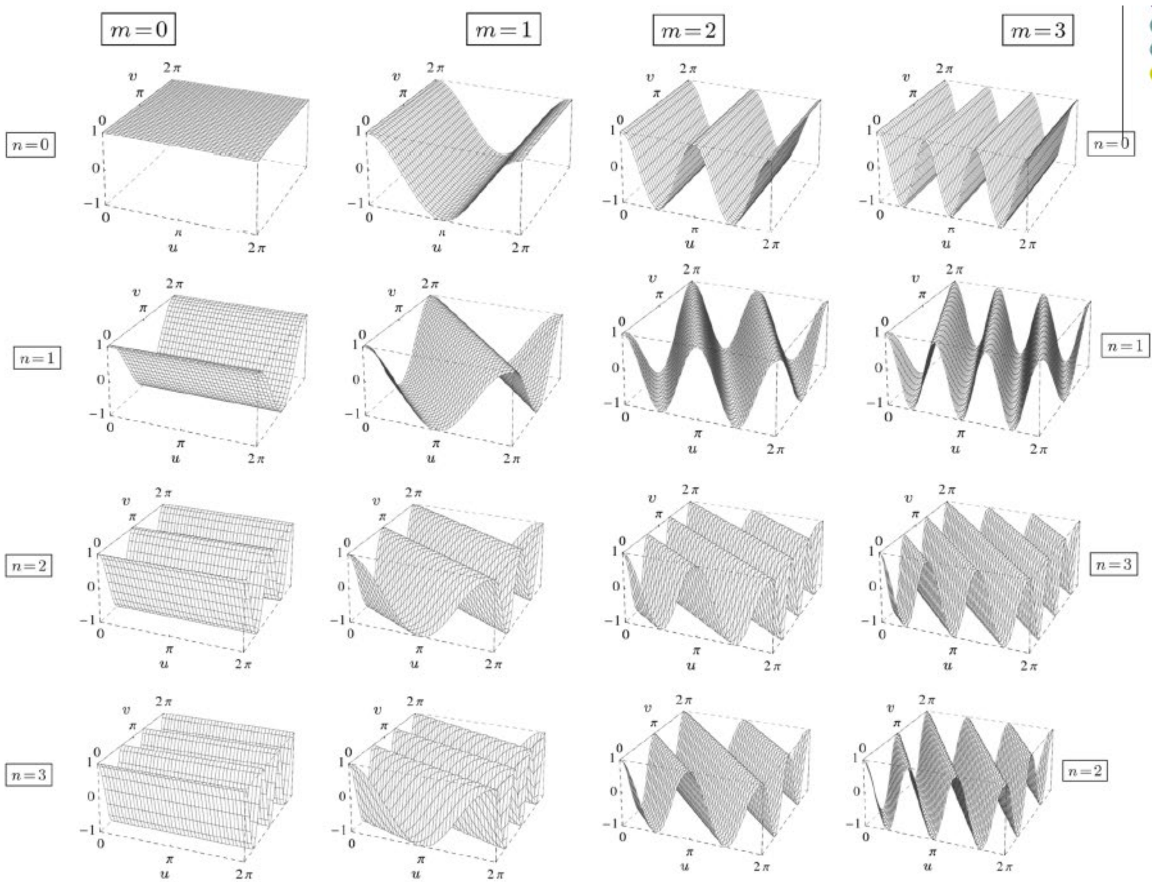
- Image can be thought of as 2D function f that can be expressed as a sum of a sines and cosines along 2 dimensions

$$e^{\mathrm{i}2\pi(\frac{mu}{M}+\frac{nv}{N})} = e^{\mathrm{i}(\omega_m u+\omega_n v)}$$
$$= \underbrace{\cos\left[2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right]}_{\boldsymbol{C}^{M,N}_{m,n}(u,v)} + \mathrm{i}\cdot\underbrace{\sin\left[2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)\right]}_{\boldsymbol{S}^{M,N}_{m,n}(u,v)}$$

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp\left[-2\pi i\left(\frac{xu}{M}+\frac{yv}{N}\right)\right].$$
$$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\exp\left[2\pi i\left(\frac{xu}{M}+\frac{yv}{N}\right)\right].$$

- ***x*** indices go from ***0… M – 1*** (x cycles over distance ***M***)
- ***y*** indices go from ***0… N – 1*** (y cycles over distance ***N***)
- All properties of 1D Fourier transform apply + additional properties

2D Cosines Functions



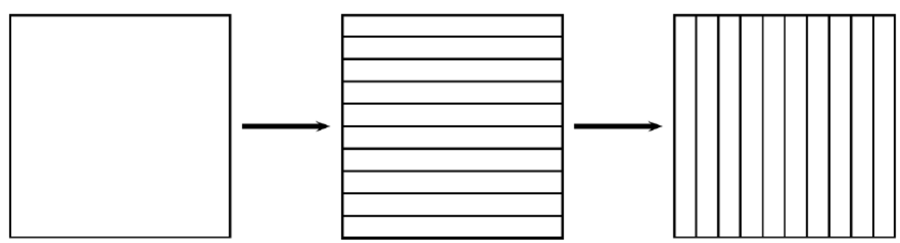
Properties of 2D DFT

- **Separability**

$$\exp\left[2\pi i\left(\frac{xu}{M}+\frac{yv}{N}\right)\right]=\exp\left[2\pi i\frac{xu}{M}\right]\,\exp\left[2\pi i\frac{yv}{N}\right]$$

2D DFT 1D DFT (row) 1D DFT (column)

- Using their separability property, can use 1D DFTs to calculate rows then columns of 2D Fourier Transform



(a) Original image (b) DFT of each row of (a) (c) DFT of each column of (b)

Properties of 2D DFT

- **Rotation**

  - Let $F(\mu,v)$ denote the Fourier transform of $f(x,y)$, then the (2D) Fourier rotation theorem says that the Fourier transform of a rotated function $f(x\cos\theta+y\sin\theta,-x\sin\theta+y\cos\theta)$ is $F(\mu\cos\theta+v\sin\theta,-\mu\sin\theta+v\cos\theta)$

  - $F(\mu\cos\theta+v\sin\theta,-\mu\sin\theta+v\cos\theta)$ is the rotated version of $F(\mu,v)$ by the same angle θ.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

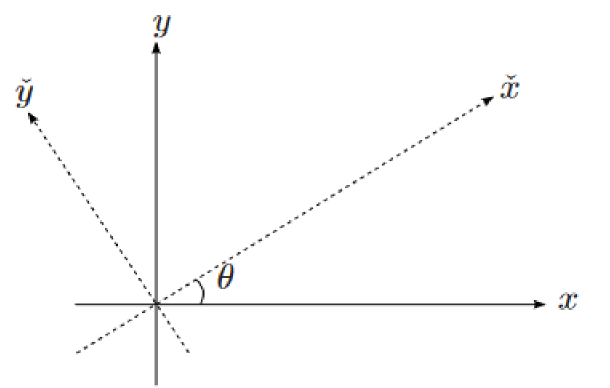
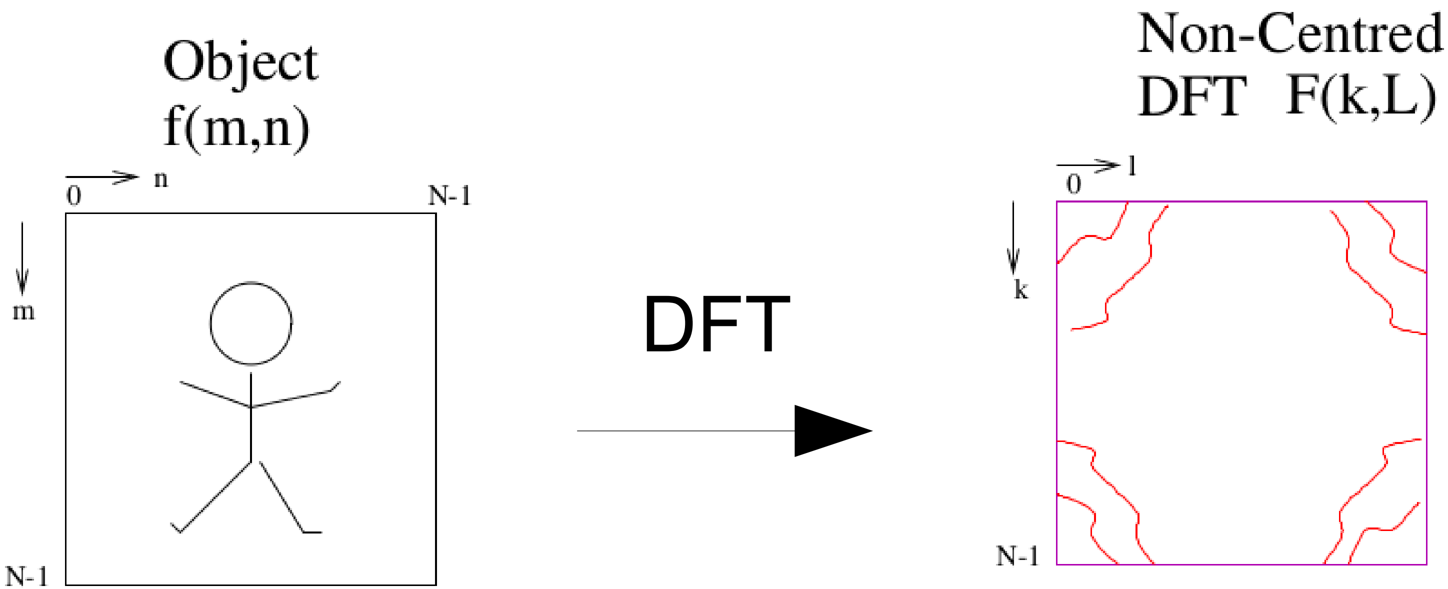
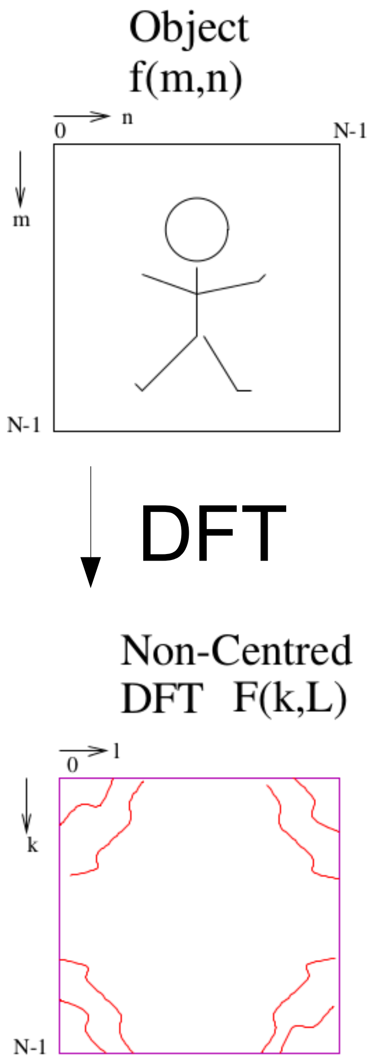


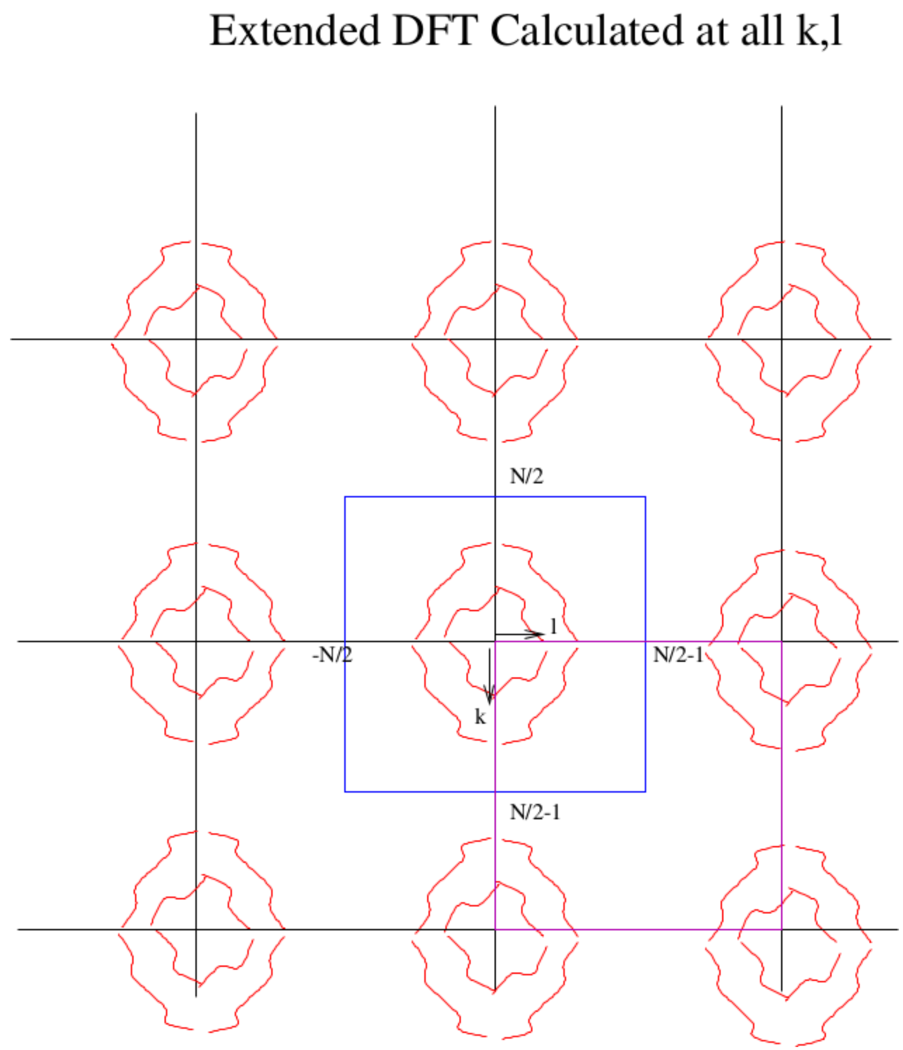
Illustration of a rotation in coordinates.

Centering: Looking at DFTs

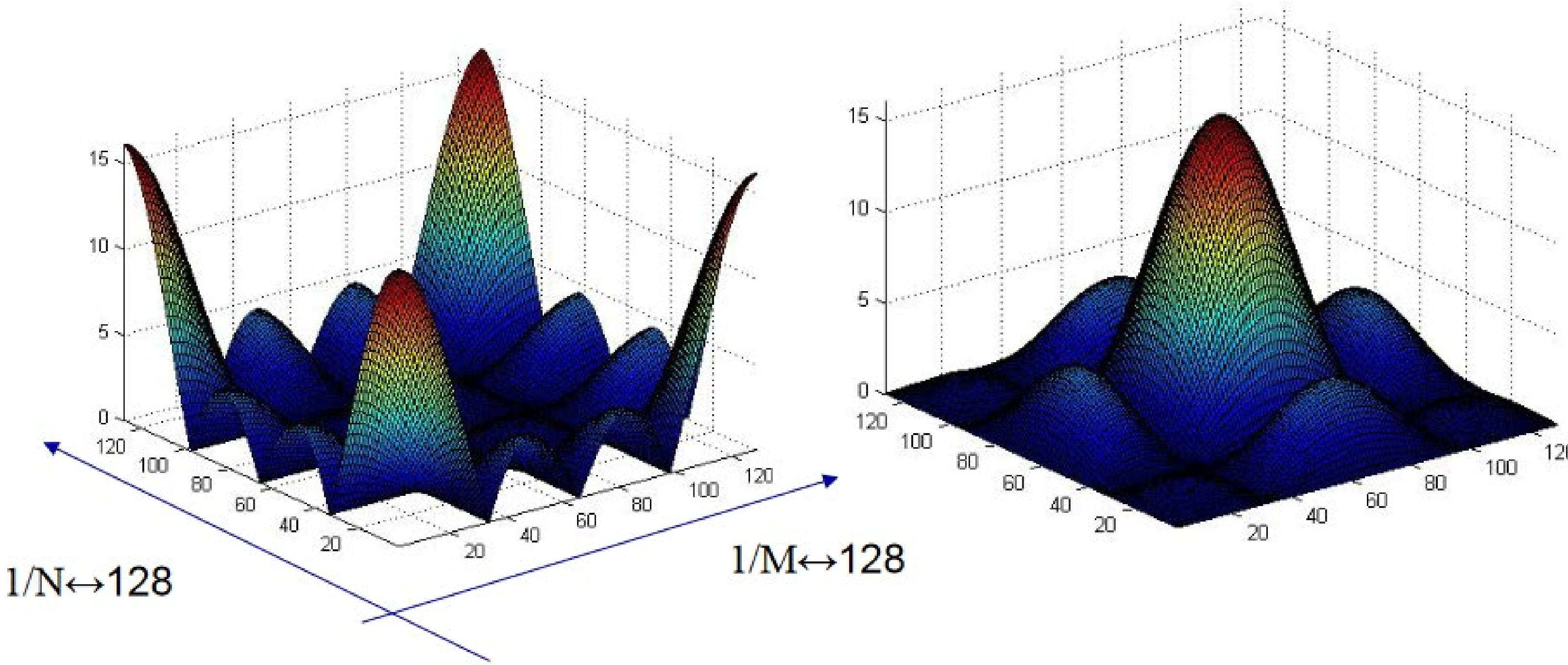


Centering: Looking at DFTs



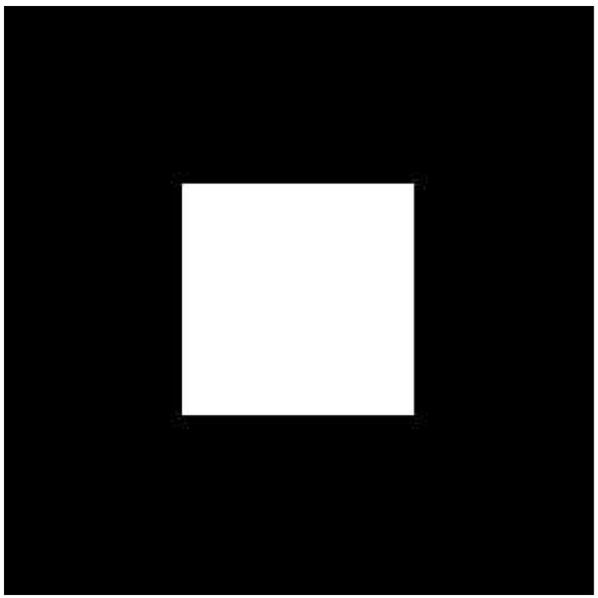


2D DFT Shift

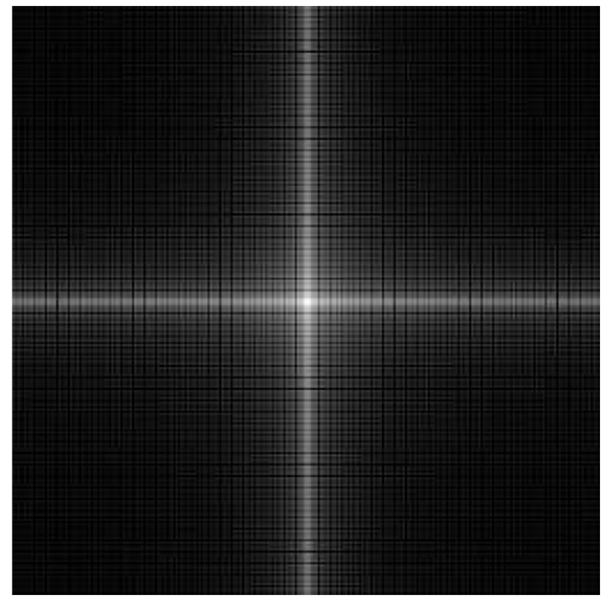


15
10
5
0
120
100
80
60
40
20
20
40
60
80
100
120
1/N↔128
1/M↔128
15
10
5
0
120
100
80
60
40
20
20
40
60
80
100
120

FT Example: A Box

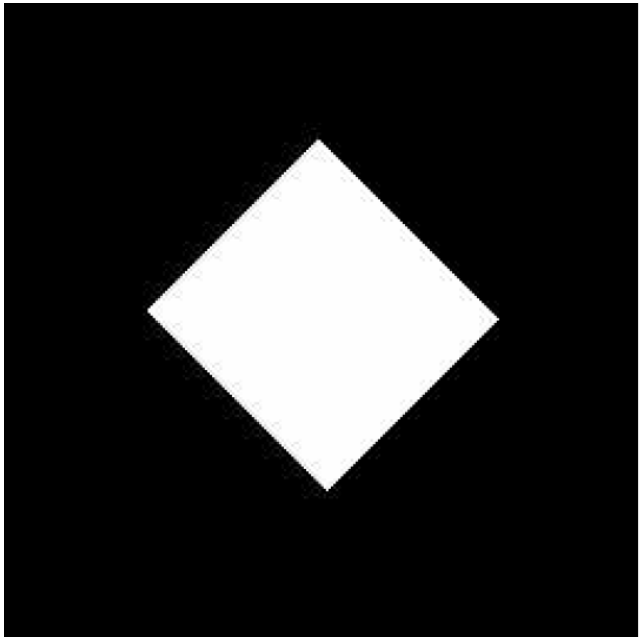


Box

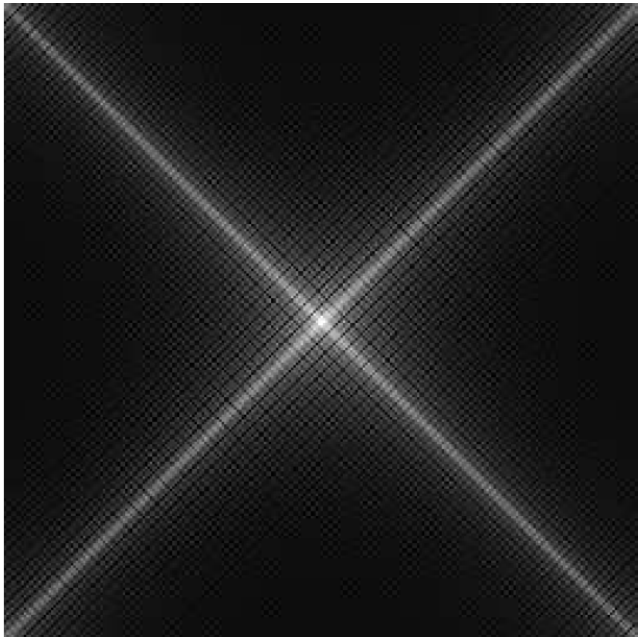


FT

FT Example: Rotated Box



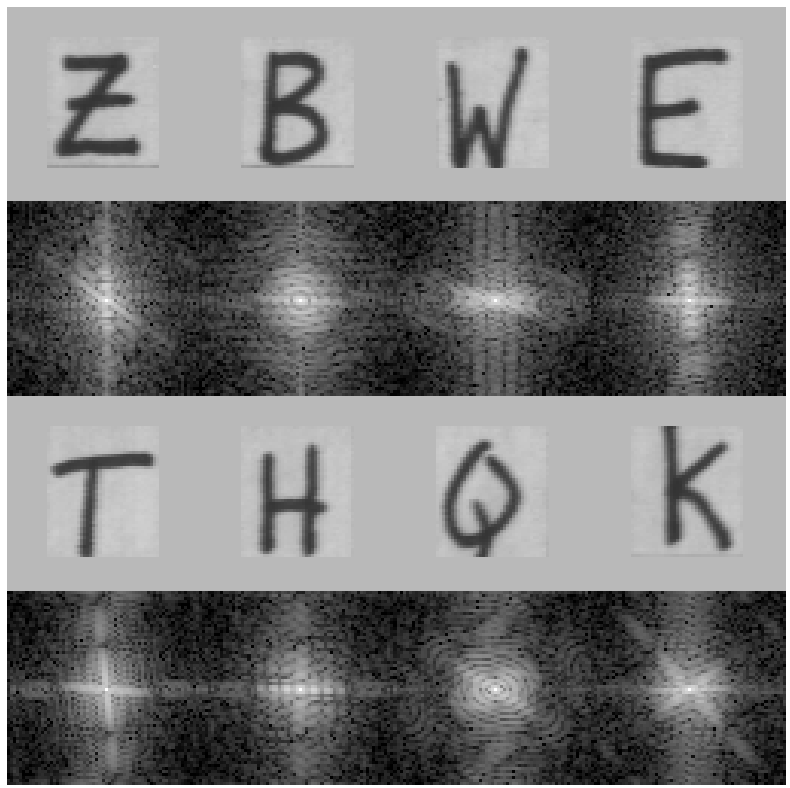
Rotated Box



FT

FT Example: Lines

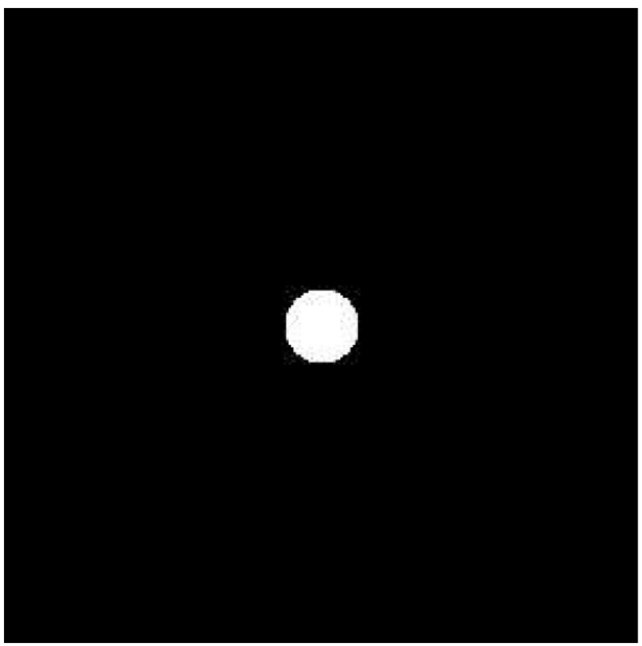
The FTs also tend to have bright lines that are perpendicular to lines in the original letter. If the letter has circular segments, then so does the FT.



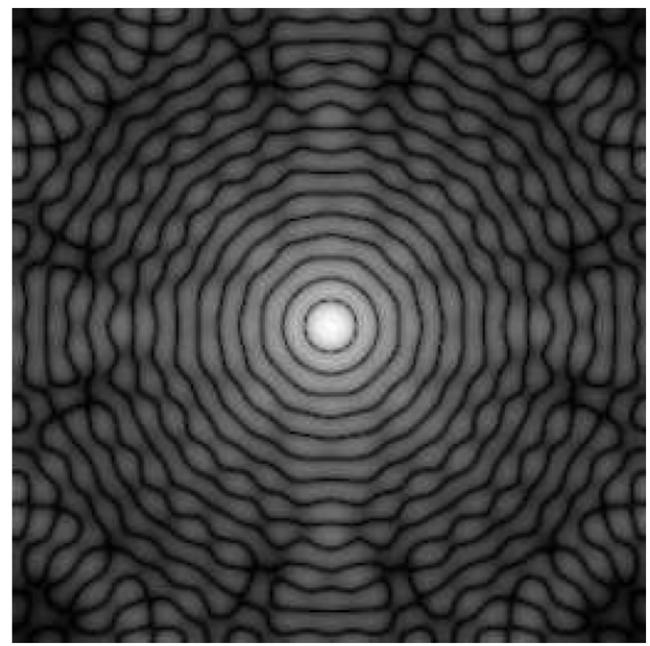
FT Example: A Circle

Note: Ringing caused by sharp cutoff of circle

Ringing does not occur if circle cutoff is gentle



Circle

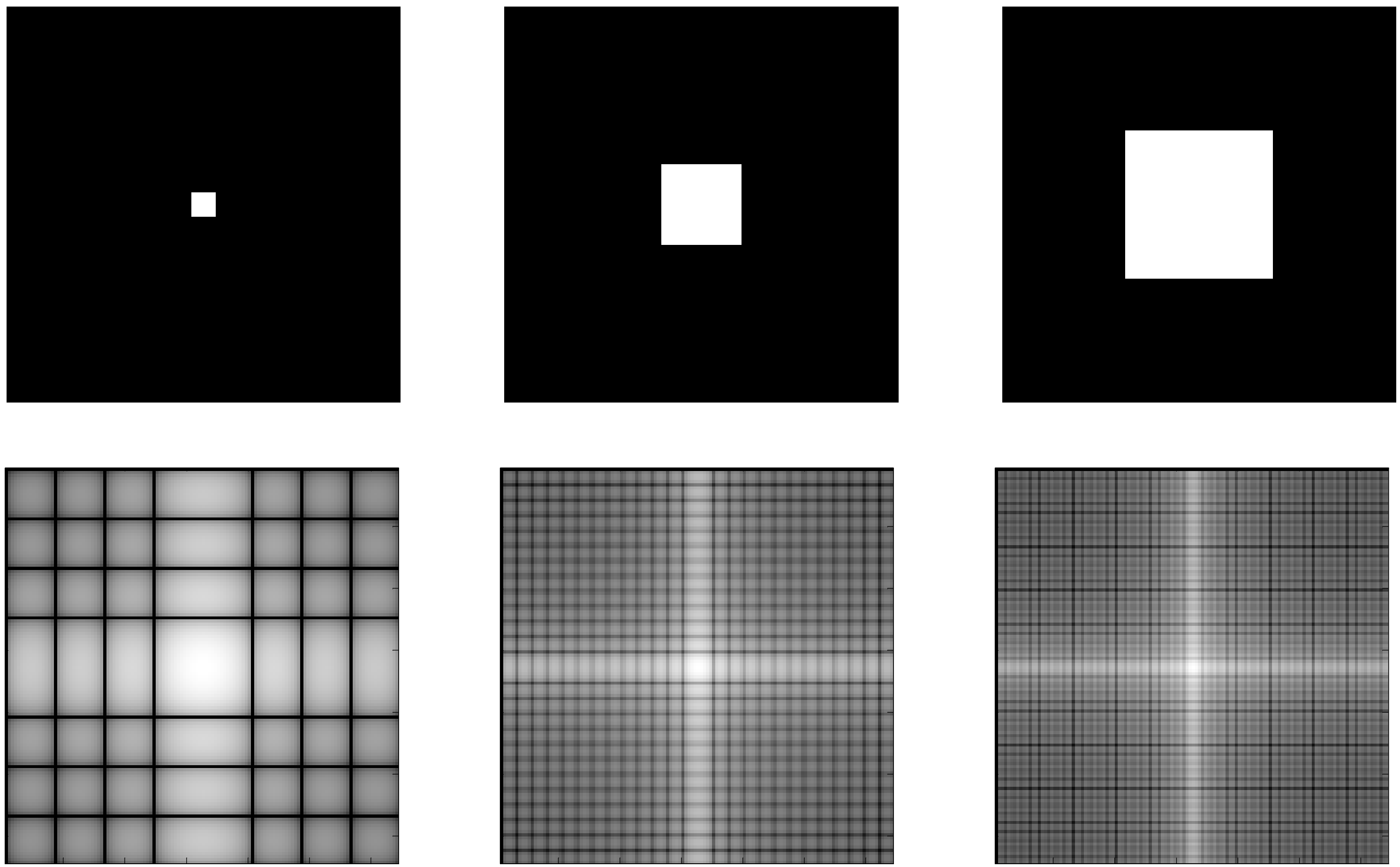


FT

2D Fourier Transform Examples: Scaling

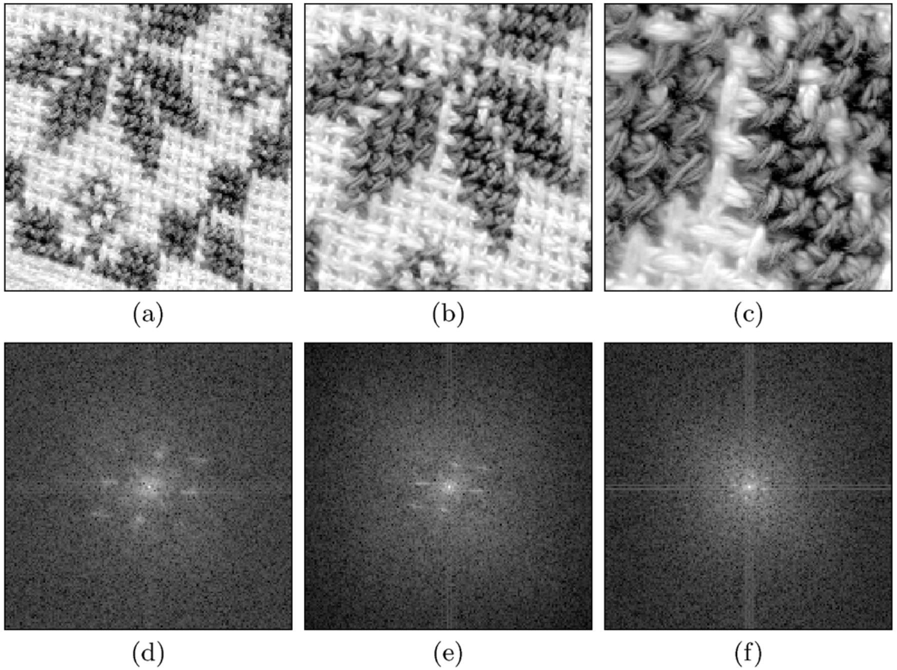
Stretching image => Spectrum contracts

And vice versa



2D Fourier Transform Examples: Periodic Patterns

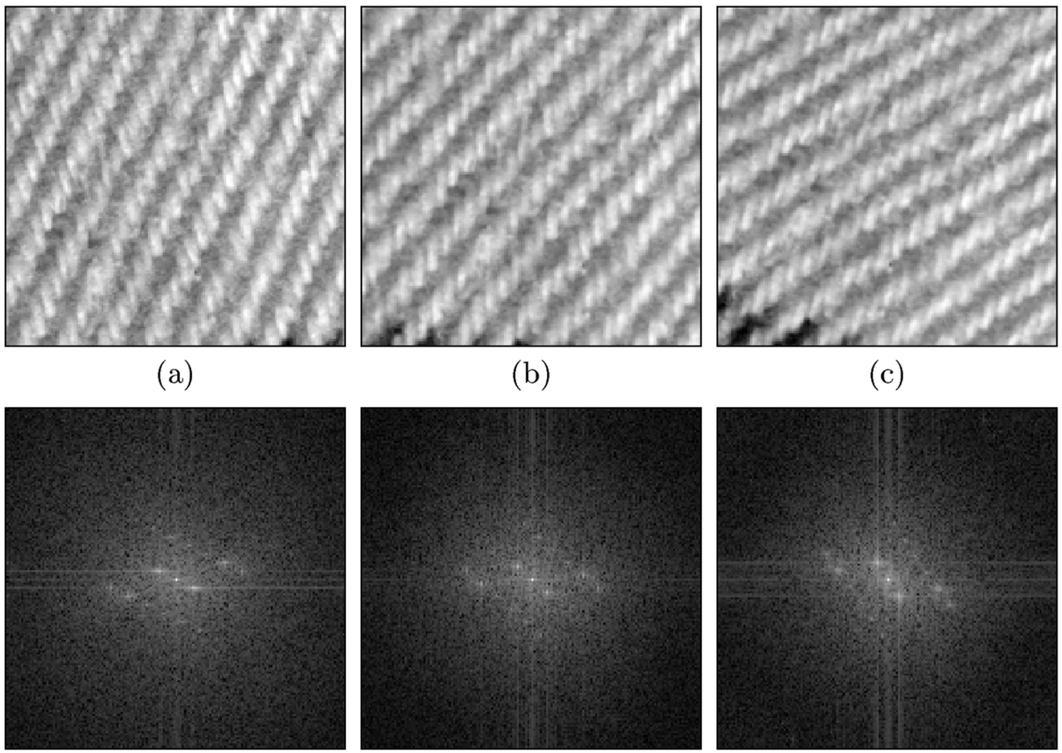
Repetitive periodic patterns appear as distinct peaks at corresponding positions in spectrum



Enlarging image (c) causes Spectrum to contract (f)

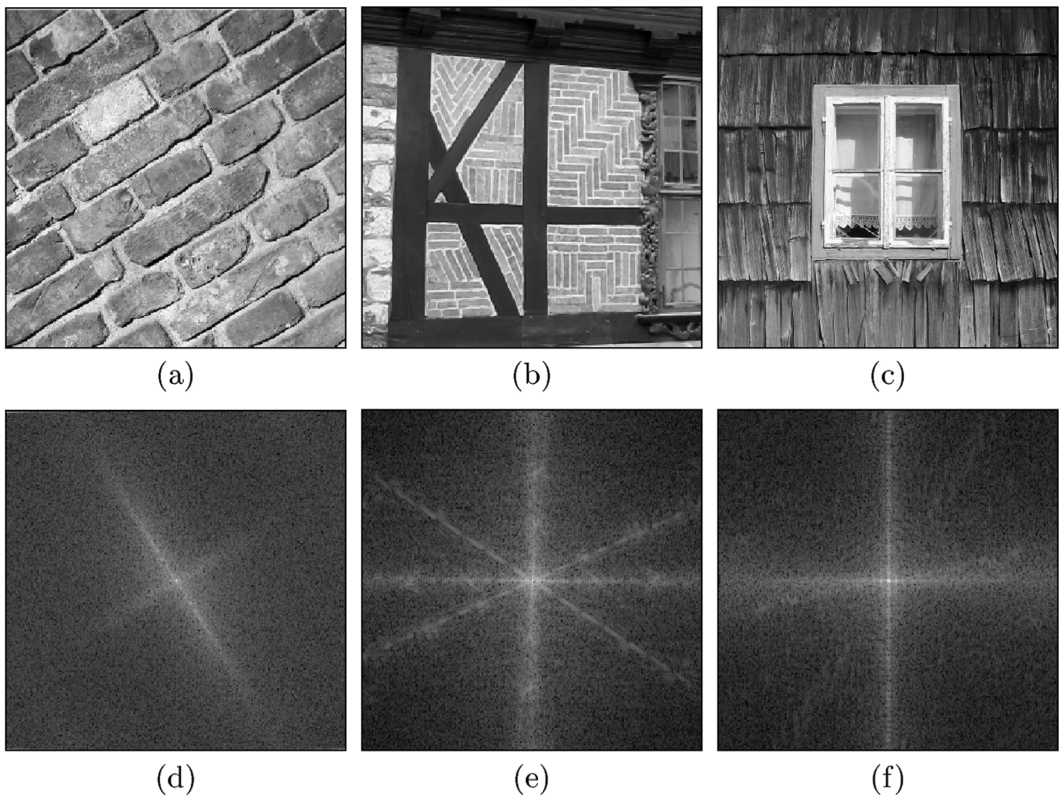
2D Fourier Transform Examples: Rotation

Rotating image => Rotates spectra by same angle/amount


(a) (b) (c) (d) (e) (f)

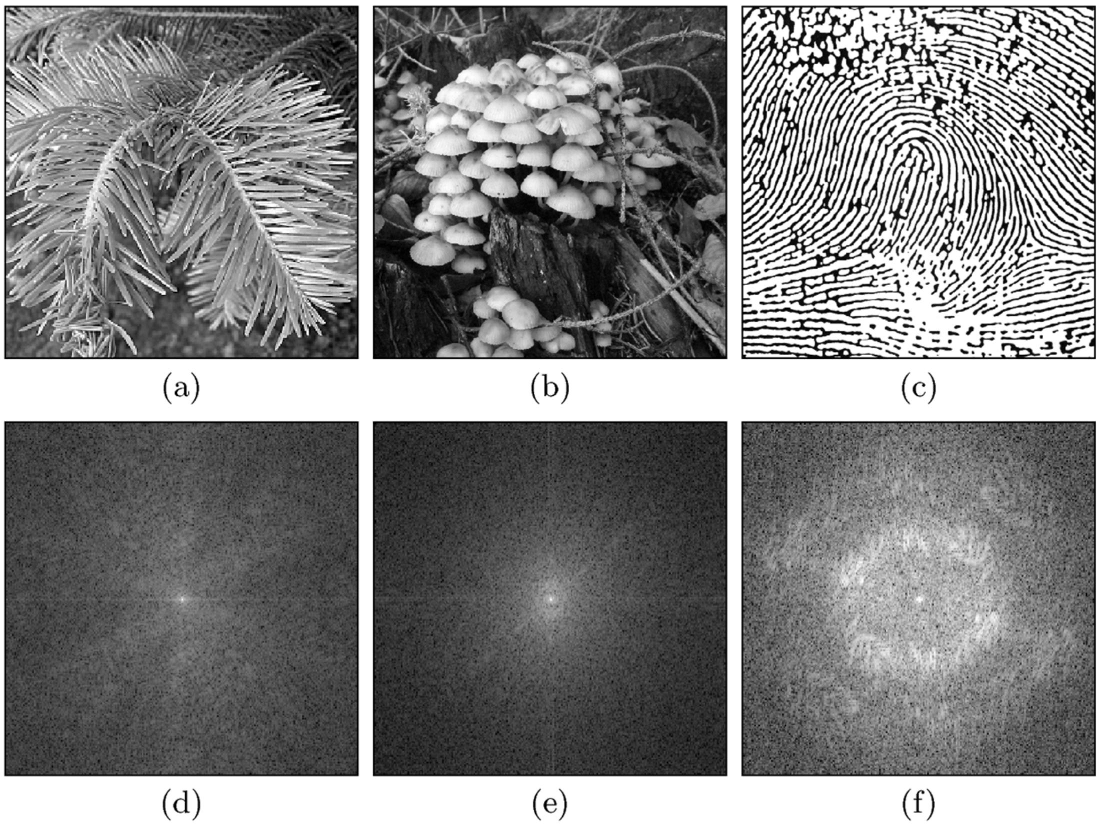
2D Fourier Transform Examples: oriented, elongated structures

Man-made elongated regular patterns in image => appear dominant in spectrum


(a) (b) (c) (d) (e) (f)

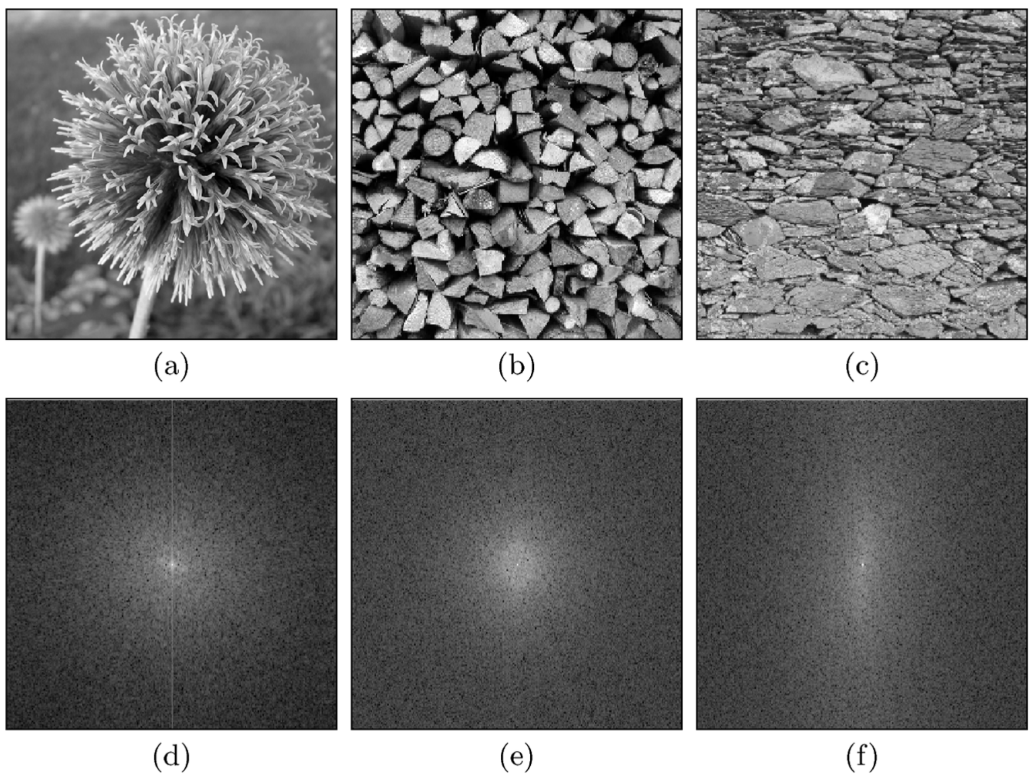
2D Fourier Transform Examples: Natural Images

Repetitions in natural scenes => less dominant than man-made ones, less obvious in spectra



2D Fourier Transform Examples: Natural Images

Natural scenes with repetitive patterns but no dominant orientation => do not stand out in spectra



2D Fourier Transform: Convolution Theorem

- FT provides alternate method to do convolution of image M with spatial filter S

1. Pad S to make it same size as M, yielding S'
2. Form FTs of both M and S'
3. Multiply M and S' element by element $\mathcal{F}(M) \cdot \mathcal{F}(S')$
4. Take inverse transform of result $\mathcal{F}^{-1}(\mathcal{F}(M) \cdot \mathcal{F}(S'))$.

$$M * S = \mathcal{F}^{-1}(\mathcal{F}(M) \cdot \mathcal{F}(S'))$$

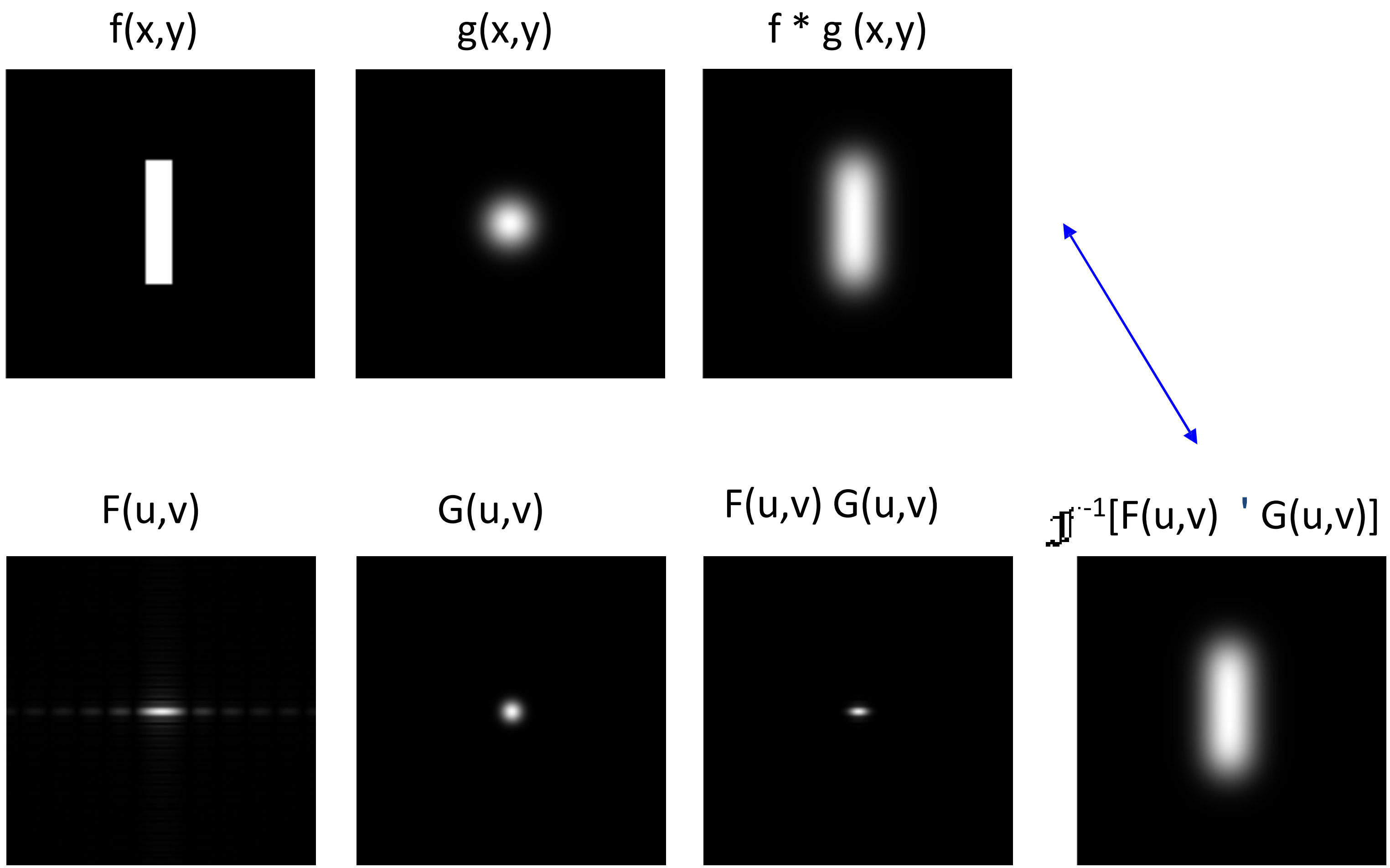
$$\mathcal{F}(M * S) = \mathcal{F}(M) \cdot \mathcal{F}(S')$$

2D Fourier Transform: Convolution Theorem

- A general linear convolution of $N_1 \ x \ N_1$ image with $N_2 \ x \ N_2$ convolving function (e.g., smoothing filter) requires in the image domain of order $N_1^2 N_2^2$ operations.
- Instead using DFT, multiplication, inverse DFT one needs $4N^2 \log(2N)$operations.
  Here N is the smallest 2^n number greater or equal to $N_1 + N_2 - 1$.

When you don’t want to do a convolution

2D Fourier Transform: Convolution Theorem



References

- Digital Image Processing (CS/ECE 545) Lecture 10: Discrete Fourier Transform, Prof Emmanuel Agu
- Lecture 2: 2D Fourier transforms and applications, B14 Image Analysis Michaelmas 2014 A. Zisserman
- EE 524, Fall 2004, # 5 http://home.eng.iastate.edu/~julied/classes/ee524/LectureNotes/l5.pdf
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